A UNIVERSAL DEPARTURE FROM THE CLASSICAL PERIOD DOUBLING SPECTRUM

D.L. GONZALEZ*, M.O. MAGNASCO** and G.B. MINDLIN***
Laboratorio de Fisica Teorica, Departamento de Fisica, U.N.L.P., CC 67, (1900) La Plata, Argentina

H.A. LARRONDO
Facultad de Ingenieria, J.B. Justo y Pampa, (7600) Mar del Plata, Argentina

L. ROMANELLI
CAERCEM, Julian Alvarez 1218, (1414) Buenos Aires, Argentina

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The topology of the period doubling attractor at the onset of chaos, and its implications for the structure of the power spectrum are discussed. The presence of a seemingly anomalous peak at a non $2^n$ frequency is explained in terms of a topological invariant of the attractor whose systematics is shown to be universal. An experiment on an electronic circuit is performed, and its results are shown to be fully consistent with the theory.

1. Introduction

Deterministic chaos arises in many branches of science [1], from classical dynamics to ecological systems or modulated lasers. One of the most noticeable features of a chaotic solution of a deterministic system is the presence of a wide band Fourier spectrum. This has led Fourier analysis to be a widely used tool, since it allows one not only to see the presence of chaos, but also to understand the structure of the chaotic attractor. This is especially true in experimental systems, for the power spectrum is easy to monitor online in real time. It is therefore important to know which factors affect the shape and structure of this spectrum. We will center our discussion on dissipative systems presenting a period doubling (Feigenbaum) route to chaos and, in particular, on the onset of chaos in such cascades.

It is the purpose of this work to enlighten the role played by what we call topological frequencies in the organization of the attractors in phase space, and the way this affects the power spectrum, for the period doubling cascade. We present a derivation of the systematics of these frequencies, which is universal for systems modelled by unimodal maps. Furthermore, we show that in some systems this peculiar systematics may result in a 'false peak' being seen, due to an accumulation of frequencies, when a finite resolution Fourier transform is used.

This paper is organized as follows. In section 2, we review some features of the topology of the period doubling bifurcation. This is studied in terms of a topology invariant whose systematics

*Current Address: Sezione di Cinematografia Scientifica, Consiglio Nazionale delle Ricerche, Via dell'Infemo 5, 40126 Bologna, Italy.
**Current Address: The James Franck Institute, The University of Chicago, 5640 S. Ellis, Chicago, IL 60637, USA.
***Current Address: Department of Physics and Atmospheric Sciences, Drexel University, Philadelphia, PA 19104.
has been conjectured by previous numerical calculations on particular systems. In section 3 we prove that this systematics is universal for unimodal maps. The proof is based on symbolic dynamics tools, and some demonstrations are deferred to the appendix. In section 4 a model of period doubling attractor which respects the topological features already discussed is presented. The consequences of this modelization and systematics for the power spectrum are treated. In section 5 an experiment on an electronic circuit is reported. It shows full agreement with the discussions mentioned above.

2. Topological and metrical information

Whenever the dynamical behaviour of a system is studied, it is desirable to make a characterization in terms of quantities that do not change under a smooth coordinate transformation of phase space. This is so because, in experimental systems, one never knows which are the dynamical variables of interest, or the whole phase space in which the relevant dynamics does occur. The properties that do not change under smooth transformations include metric, dynamical, structural and topological information. Much work has been done in order to show the presence of universalities in these properties for various classes of systems. Metric information includes the scaling indexes and multifractal spectra, and depends on the geometrical properties of the attracting set. Dynamical information includes the Lyapunov exponents and the entropy spectra, and depends on the properties of trajectories [2]. Please note that in this sense, the power spectrum is not metric or dynamical information, since smooth nonlinear transformations of phase space affect the spectrum (by the presence of combination tones, for instance). This, however, does not diminish the importance of the power spectrum as an experimental and theoretical tool. Metric information remains in these spectra, as shown, for instance, by Feigenbaum [3]. Structural information also leaves its mark on the spectra. In particular, we will show how a large class of systems preserve some topological features.

We will center our discussion on the topological characteristics (they are structural properties) and we study the specific case of dissipative systems undergoing a period doubling cascade.

In these systems, the attracting set of a chaotic solution is knotted in a complex way. As we will show in this paper the frequency components needed for the intrinsic construction of the knot have internal self-similarity properties.

The spatial structure of systems undergoing period doubling has already been studied [4–7]. Numerical investigations of the invariant manifolds of the periodic attractors revealed their twisted nature, and the systematics ruling the number of twists of these manifolds as a system parameter is varied has been stated for particular systems. Let us review this systematics.

We define the bidimensional stable manifold of a certain stable or marginally stable periodic solution as the strip swept by the eigenvector corresponding to the largest eigenvalue of the evolution operator, linearized around this solution. Note that the largest eigenvalue is $< 1$ because we are dealing with attractors.

That is, for a system $\frac{dx}{dt} = f(x)$ (where $\mu$ is a parameter ruling the period doubling cascade) with a stable $T_n$ periodic solution $x^*(t)$ we say that $x(t)$ belongs to the bidimensional stable manifold $W_s$ if

$$x(t) = k (T e^{itDf_\mu} \omega),$$

(1)

where $\omega$ is the eigenvector corresponding to the maximal eigenvalue of

$$T e^{itDf_\mu}.$$

$T$ is the time order operator, $Df_\mu$ is the Jacobian matrix and $|k| \ll 1$.

In fig. 1 we show schematically how the bidimensional stable manifold evolves as the system goes on a period doubling bifurcation.
When the periodic attractor is born the bidimensional stable manifold has the form of an orientable strip (fig. 1(a)); as the parameter varies, an odd number of half turns is added (or subtracted) in a region with complex eigenvalues and when the parameter \( \mu \) has a value close to \( \mu_c \) (the critical value at which the bifurcation takes place), the bidimensional stable manifold is twisted around the periodic orbit as a non-orientable strip (the Möbius strip of fig. 1(b)). The eigenvalue will be close to \(-1\), and in principle it is only known that the number of half turns will be odd, but the exact number can only be calculated through simultaneous integration of the full and linearized equations of motion. This number is very important since it characterizes not only the local structure of the flow, but also the knot type of the new periodic solution that will be born when the eigenvalue passes through \(-1\) and the central solution loses its stability.

It is important to understand why the strip should be non-orientable just before a bifurcation. When the eigenvalue reaches a value of \(-1\), a perturbation will also be, up to first order, a periodic solution. Therefore, the bifurcation will be a period doubling, since the perturbation has to travel along the central solution twice. When the eigenvalue has just passed through \(-1\), the old central manifold will be a repeller, and the stability would have been transferred to the edges of the strip. In this sense, the topology of perturbations before the bifurcation determines the topology of the solution after the bifurcation (see fig. 1(c)).

Let us define, for a periodic solution, a topological number \( g \) as the quotient between the number of twists of the invariant manifold and the period of the solution. As the parameter ruling the period doubling cascade is varied, we obtain for a period \( 2^i \), \((i = 0, 1, \ldots)\) at least two values of \( g \): one corresponding to the value of \( \mu \) at which the solution is born, and the other corresponding to the value of \( \mu \) at which a new bifurcation is going to occur. That is,

\[
g_i^0 = \frac{2k}{2^i} \quad \text{(orientable case)},
\]

\[
g_i^n = \frac{2k^i + 1}{2^i} \quad \text{(non-orientable case)},
\]

with \( k \) and \( k^i \in \mathbb{Z} \).

At \( \mu_c \) the center manifold is cut, and the value of \( g \) is

\[
g_{i+1} = g_i^n = \frac{2(2k^i + 1)}{2^i} = \frac{2k^{i+1}}{2^{i+1}}.
\]

The numerical investigations for the Brusselator [4], exactly solvable model (Pirogon) [7] and driven laser [8] gave the values of \( g_i \) listed in table I. It may be seen that \( g_{i+1} = \frac{1}{2} \left[ g_i^n + g_i^0 \right] \) for all \( i \) obtained, or that \( g_i^n \) is the best dyadic approximant to \( \frac{1}{2} \) with denominator \( 2^i \) (this is valid for the case \( g_0^0 = 1 \) and \( g_0^n = \frac{1}{2} \)).
3. Symbolic dynamics

The Poincaré map connects any \( n \)-dimensional flow with an \( (n-1) \)-dimensional recurrence. In some dissipative systems each iteration contracts the phase space, and asymptotically any infinitesimal volume element evolves toward a one-dimensional manifold. This manifold is generated locally in the direction with slower convergence. In this case the Poincaré map can be approximated by means of a one-dimensional recurrence:

\[
x_{n+1} = f_\lambda(x_n),
\]

where \( \lambda \) is the control parameter. This one-dimensional map is unimodal if the function \( f_\lambda \) is continuous, with only one maximum and it has no inflection points in the interval of interest.

In the following we will use the tools of symbolic dynamics in order to show that the systematics of the bidimensional stable manifold previously mentioned holds for all \( n \), and is universal for any flow such that its return map can be modelled by a one-dimensional unimodal map.

When a one-dimensional recurrence is analyzed, the stability of a certain solution is studied in terms of the evolution of perturbations around it. This is usually done as follows:

Let \( x_0^* + \epsilon_0 \) be a point belonging to the solution. Up to first order, the evolution of an initial condition \( x_0^* + \epsilon_0 \) close to \( x_0^* \), can be written as:

\[
x_{n+1}^* + \epsilon_{n+1} = f_\lambda(x_n^* + \epsilon_n)
\]

\[
\equiv x_{n+1} + \frac{\partial f_\lambda}{\partial x} |_{x_n^*} \epsilon_n.
\]

It implies

\[
\epsilon_{n+1} = \frac{\partial f_\lambda}{\partial x} |_{x_n^*} \epsilon_n.
\]

The geometric mean of the absolute value of \( \frac{\partial f_\lambda}{\partial x} |_{x_n^*} \) is called the stability parameter; its logarithm, the Lyapunov exponent, and it determines the stability of the orbit.

The sequence \( \eta_n = \text{sgn} \left( \frac{\partial f_\lambda}{\partial x} |_{x_n^*} \right) \) of the signs of the derivatives contains an entirely different kind of information. It describes essentially topological features, and is called symbolic dynamics. If we assign a letter to each monotone segment of the function \( f_\lambda \) each orbit has a corresponding symbolic sequence.

Temporal evolution of the orbit then becomes a shift operation on a symbolic sequence.

For unimodal maps the symbolic alphabet consists of two letters. Following tradition we will use \( L \) for \( \eta = +1 \) and \( R \) for \( \eta = -1 \). The symbolic algebra for unimodal maps has been extensively studied [9], and we will use some previous results in the following part.

Note that each time an iterate \( x_n \) belongs to the \( R \) interval, the sign of \( \epsilon_n \) will be reversed in the next iteration. This is equivalent, in the flow, to the situation where, between intersections with the Poincaré surface, the tangent manifold rotates an odd number of half turns (see fig. 2(a)). Conversely, if the iterate belongs to the \( L \) interval, the tangent manifold rotates an even number of half turns (see fig. 2(b)).

We will now discuss the period doubling cascade. We will assume hereafter that the sequences are built starting from \( x_0 \), the maximum of the map. The symbolic sequence of the transient is the same as that of the stationary state.

Period 1 is born orientable. The total number of turns in the manifold is even and the symbolic sequence is an \( L \) (see fig. 3(a)). When the parame-

Table I

<table>
<thead>
<tr>
<th>Order</th>
<th>Orientable manifold</th>
<th>Non-orientable manifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2:1</td>
<td>1:1</td>
</tr>
<tr>
<td>1</td>
<td>2:2</td>
<td>3:2</td>
</tr>
<tr>
<td>2</td>
<td>6:4</td>
<td>5:4</td>
</tr>
<tr>
<td>3</td>
<td>10:8</td>
<td>11:8</td>
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<tr>
<td>4</td>
<td>22:16</td>
<td>21:16</td>
</tr>
<tr>
<td>5</td>
<td>42:32</td>
<td>43:32</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
D. L. Gonzalez et al. / Departure from classical period doubling spectrum

Fig. 2. Poincaré map for the flow of fig 1: (a) $\mu < \mu_c$: $x^*$ is the intersection of the periodic solution of period $T$ with the Poincaré section; (b) $\mu \leq \mu_c$: note that $t_{n+1}$ and $t_n$ are antiparallel; (c) $\mu \geq \mu_c$: now we have a solution with period $2T$.

ter is increased, the tangent manifold becomes non-orientable (preparing for the bifurcation). By the above argument, the sequence is an $R$ (see fig. 3(b)). Upon doubling, the orbit is essentially the same, travelled along twice. So the sequence is $RR$ (see fig. 3(c)), and is orientable, as we expected. Increasing the parameter again, the manifold becomes non-orientable in preparation for the second doubling. The sequence is then $RL$. The process goes on in the same way: the non-orientable manifold doubles copying the former sequence twice, then this orientable manifold becomes non-orientable by changing the last letter in the sequence. The letter to be changed must always be the last one, since in passing from $L$ to $R$ the maximum is in between; if a former letter is

Fig. 3. Symbolic sequences corresponding to a dissipative flow. The Poincaré map is asymptotically a one-dimensional map (it is assumed to be unimodal). (a) $\mu < \mu_c$: the sequence is an $R$; (b) $\mu \leq \mu_c$: the sequence is an $L$; (c) $\mu \geq \mu_c$: the sequence is $RR$. 

Table II
The symbolic sequences corresponding to the first bifurcations.

<table>
<thead>
<tr>
<th>Order</th>
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<th>g</th>
<th>Non-orientable manifold</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>L</td>
<td>2:1</td>
<td>R</td>
<td>1:1</td>
</tr>
<tr>
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<td>RR</td>
<td>2:2</td>
<td>RL</td>
<td>3:2</td>
</tr>
<tr>
<td>2</td>
<td>RRLR</td>
<td>6:4</td>
<td>RLLR</td>
<td>5:4</td>
</tr>
<tr>
<td>3</td>
<td>RRRRLLLRRR</td>
<td>10:8</td>
<td>RRRRLLLRLR</td>
<td>13:8</td>
</tr>
<tr>
<td>4</td>
<td>RRRRLLLRLRLLLRLRLRL</td>
<td>22:16</td>
<td>RRRRLLLRLRLLLRLRRLRLRL</td>
<td>21:16</td>
</tr>
</tbody>
</table>

changed, there would be another period in the middle. Table II lists the sequences corresponding to the first bifurcations. Note that the number of R’s in the non-orientable sequence follow the numerology of table I. By continuity, if the perturbation around an \( x_n \) in the \( R \) interval rotates an odd number \( k \) of half turns before arriving at \( x_{n+1} \), then \( k \) will be constant on all of the \( R \) intervals. The same applies to \( L \). As there is no global bifurcation, these values are constant also along \( \lambda \).

So, the value of \( R \) will always be \( g_0^n \), and \( L \) will be \( g_{-1}^n \). For three of the four references cited, \( R = 1 \), \( L = 2 \) holds (this is always so for self-oscillators with only one stable limit cycle, and a fixed point repeller, like the van der Pol model, in the regime of strong relaxation).

When an infinite number of doublings has just taken place, we are at the onset of chaos. (The Lyapunov exponent is zero, so the set is not really chaotic.)

![Fig. 4. The Fourier spectrum of the number theoretical Morse–Thue sequence.](image-url)
The symbolic sequence for the onset of chaos has the following renormalization property: the sequence is invariant under a substitution of the form $R \rightarrow RL$, $L \rightarrow RR$. Using this property we show in appendix A that the density of $R$'s in the sequence is $2/3$.

With the above assignment $(R \rightarrow 1, L \rightarrow 2)$, $g$ is equal to $2/3$ (of $2\pi$). This value is universal under the conditions stated above.

The symbolic sequence $\eta$ represents the signs of the derivative of the map at the points of the orbit. The signs of the perturbations themselves are given by

$$x_{i+1} = x_i \eta_i = \prod_{k=1}^{i} \eta_i \quad (x_0 = 1). \quad (3)$$

In appendix B we show that the sequence $\xi$ is the number theoretical Morse–Thue sequence [10]. The power spectrum for this sequence is shown in fig. 4. This sequence is a simplified model of the evolution of perturbations around the period doubling attractor. The fact that this sequence has a peak at $2/3$ reflects the fact that perturbations, in the limit, contribute to the spectrum with components near $2/3$. The sequence has also a noteworthy property: its spectrum is orthogonal to the spectrum of the attractor, since it is zero for any dyadic rational.

4. Flow and map spectra

The classical Feigenbaum spectrum [3] was derived for the stationary state of a unimodal map at the onset of chaos. It presents components at and only at dyadic rational frequencies, that is, frequencies of the form $m \cdot 2^{-n} \omega_0$, $n \in \mathbb{N}$. The main universality result is that the amplitudes of $2^n$ frequencies are scaled with that of nearby $2^{n+1}$ frequencies through $a$, the universal scaling factor.

However, almost all experimental power spectra are flow power spectra. There are differences between the two, since in strobing the flow with the Poincaré section we lose information on the shape of the flow, which affects the spectrum. The former is a discrete Fourier transform, the latter a continuous Fourier transform. In the map spectra, all harmonics over the fundamental frequency are lost.

Among the frequency components of the spectrum, we will distinguish the topological frequencies, that is, frequencies that must be added in order to form the right kind of knot. These frequencies form the topological skeleton of the attractor. They show universal behaviour. The other frequencies are not, we stress again, metric information, and depend strongly on the particularities of the flow.

We claim that an $x(t)$ defined as

$$x(t) = \sum_{i=0}^{\infty} \epsilon_i \cos \left( g_i^m t + \varphi_i \right)$$

$$= \cos \left( t + \varphi_0 \right) + \epsilon_1 \cos \left( \frac{1}{2} t + \varphi_1 \right)$$

$$+ \epsilon_2 \cos \left( \frac{3}{2} t + \varphi_2 \right) + \epsilon_3 \cos \left( \frac{5}{2} t + \varphi_3 \right) + \cdots \quad (4)$$

contains all topological information needed to define the knot type of the onset of chaos attractor. The $\epsilon_i$ contain metric information, and $\lim \epsilon_i/\epsilon_{i+1} \rightarrow -a$. The $\varphi_i$ are related to dynamical information such as visiting orders.

Let us take the sum up to an order $n$ instead of infinity. Then $x_n(t)$ is a $2^n$ knot, and these $x_n$ follow the right twisting systematics described before, as $n$ increases (see fig. 5). Note that $x(t)$ is a good model of a period doubling attractor, although it carries a minimum amount of structural and metric information.

The Fourier transform of $x(t)$ is obviously

$$\hat{x}(\omega) = \sum_{i=0}^{\infty} \epsilon_i e^{i\varphi} \delta(\omega - g_i^m). \quad (5)$$

Note that the spectrum of this series differs considerably from the classical period doubling spectrum, since it does not have any component for $2^l$ frequencies other than $g_l^m$. Nevertheless, if a
smooth coordinate transformation is applied to phase space, all other dyadic frequency components may be recovered, by means of combination tones. For instance, if the transformation

\[ x^d(t) = x(t) + \varepsilon [x(t)]^3 \]  

is performed, every \(2^i\) frequency is present for \(\varepsilon > 0\). See fig. 6. Moreover, they still scale with adjacent peaks through \(\varepsilon\).

As \(g_i^* \rightarrow 2/3\), there is an accumulation of dyadic peaks around \(\omega = 2/3\) in (5). If a finite resolution Fourier transform is performed on \(x(t)\), as would happen in an experimental situation, or if \(x(t)\) had been obtained numerically, a peak at \(\omega = 2/3\) would actually be seen (see fig. 7). (Note that, with

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Fig. 5. Phase space trajectories \(x_\alpha(t) = \sum_{n=1}^{\infty} (\alpha)^{-1} \sin(2\pi g_i t + \phi_i)\). The \(\phi_i\) were chosen randomly for this figure. (a) \(x_1(t)\), period 2; (b) \(x_2(t)\), period 4. The construction then proceeds along in the same way.
the above transformation, for low $\varepsilon$ the 2/3 peak would still be clear.) Some systems in experimental situations may show such an 'anomalous' frequency component, as indeed has been observed [11]. We have called such frequency components Caballi [12]. Even if such a false frequency component is not observed, it is a common feature of spectra to have $g^n$ components higher than the other components of the same denominator.

5. The experiment

In fig. 8 a monostable circuit is shown. When this system is forced by a sinusoidal signal and the voltage drop across the diode is recorded, a period doubling transition to chaos occurs as the amplitude of the driving signal is increased (see fig. 9).

This circuit is quite stable, the measurements are repeatable and do not require special precautions. We have used a standard waveform generator, which generates a sinusoidal wave of 100 kHz. The spectrum analyzer was used with a window of 100 Hz and a scan time of 20 sec. In this form it is possible to obtain a very detailed spectrum.

In fig. 9(a) it is important to note the significant difference between subharmonics 1/4 and 3/4. If the linear scale of the analyzer is employed (as in fig. 9(c)) the subharmonic 1/4 cannot be seen (the presence of a small component 1/4 is detected using logarithmic scale). Precisely 3/4 is one of the topological frequencies. The same kind of behaviour is present in all other bifurcations; this feature is responsible for the accumulation of peaks around the frequency 2/3 giving a false maximum in the flow spectrum at the onset of chaos (see fig. 10). The presence of all the other subharmonics can be explained on the basis of the argument given above. A system very similar to ours was investigated extensively [13]. A one-
dimensional map was obtained so the system belongs to the class studied by us.

6. Conclusions

The systematics of the $g$ invariant has been proved for all unimodal maps. We have shown the role of the $g_i^n$ frequencies (the topological frequencies) in organizing the structure of the period

Fig. 8. The circuit. $D = 1N4004$, $L = 30$ mH, $f = 100$ kHz.

Fig. 9. The first doublings of the circuit. Power Fourier spectrum vs. frequency. Note that the $g_i$ components are always higher than other subharmonic frequencies of the same period.
Doubling attractor. This has led us to present a modelization of the attractor which takes into account these features, together with a minimum account of metrical information. We have shown the effect of the accumulation of $g^n$'s when a system whose attractor is "near" our model is studied with a finite resolution Fourier transform. We have actually studied experimentally such a system, and shown that the results are fully consistent with the theory.

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Appendix A

Density of R's in the symbolic sequence corresponding to the onset of chaos

The symbolic sequence corresponding to the onset of chaos, in a one-dimensional unimodal map is

$$S = RLRRRLRLRRRLRRRRRR...,$$

defined as

$$S = R \ast \infty,$$

where the symbol $\ast$ denotes the product defined as follows:

Let $A = \{A_0, A_1, A_2, \ldots, A_n\}$ and $B = \{B_0, B_1, B_2, \ldots\}$. The product $A \ast B$ is

i) $A \ast B = A_0 A_1 A_2 \ldots$

if the number of $R$'s in $A$ is even;

ii) $A \ast B = A_0 A_1 A_2 \ldots$

if the number of $R$'s in $A$ is odd

$L = \bar{R}; R = \bar{L}$.

For the bifurcation cascade it follows:

$$R \rightarrow R \ast R = RLR$$

$$\rightarrow R \ast RLR = RLRRLR$$

$$\rightarrow R \ast RLRRLR = RLRRLRLRLR$$

and so on

$$\rightarrow R \ast \infty.$$

The sequence has a binary tree structure defined by the following discrete renormalization proper-
ties (see fig. 11(a)): 

\[ S = R \ast \infty \rightarrow \begin{cases} S = S \ast R, \\ S = R \ast S. \end{cases} \]  

Eq. (A.1) defines the binary tree structure and the fact that the tree is invariant under a doubling at the top.

Eq. (A.2) defines the property that in each step the substitution of \( R \rightarrow RL \) and \( L \rightarrow RR \) creates the next level. This can be seen as an invariance of the three under a further deepening of its levels. Note that the symbolic sequence has the same structure as the binary tree obtained connecting the last 1 of each natural number expressed in base two. See fig. 11(b). Then we can calculate the \( i \)th character in the \( S \) sequence by means of the following straightforward algorithm:

\[ S_i = (-1)^k \cdot \max \left\{ j/i \text{ mod } 2^j = 0 \right\}, \]

where we have assigned \( R = -1 \) and \( L = +1 \).

Let's take two adjacent levels in the construction of \( S \). If \( n_0^r \) and \( n_0^l \) are the number of \( R \)'s and \( L \)'s in the first level of the tree, and \( n_1^r \) and \( n_1^l \) are the same in the second level, then

\[ n_1^r = n_0^r + 2n_0^l, \]
\[ n_1^l = n_0^l. \]

It means that

\[ n_1^r/(n_0^r + n_1^l) = (n_0^r + 2n_0^l)/2(n_0^l + n_1^l) \]
\[ \rightarrow n_0^l/(n_0^l + n_1^l), \]

then

\[ n_0^l \rightarrow 2n_1^l \]

and

\[ n_0^l/(n_0^l + n_1^l) \rightarrow 2/3. \]

**Appendix B**

**The Morse–Thue sequence**

This sequence is the ‘binary root’ sequence, that is, the sum modulo two of the digits of the natural integers when expressed in base two.

\[ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ \ldots \]

\[ \rightarrow + - + - + + - - + + + + + - - - - - - - - - - - - \ldots \]

It has the following renormalization property: The sequence is invariant under decimation, which means that if every second symbol in the sequence is suppressed, the sequence is unchanged. The sequence is also invariant under the following substitution:

\[ + \rightarrow + -, \]
\[ - \rightarrow - + . \]

It has been compared with a quasicrystal [10], since it is aperiodic (by construction) but never-
theless it possesses sharp spectral peaks and high order (indeed infinite order) correlations.

References

[12] ‘Caballi: The astral bodies of men who died a premature death. “They imagine to perform bodily actions while in fact they have no physical bodies but act in their thoughts.” Paracelsus’, a footnote in Lawrence Durrell’s Justine.