Periodically kicked hard oscillators

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A model of a hard oscillator with analytic solution is presented. Its behavior under periodic kicking, for which a closed form stroboscopic map can be obtained, is studied. It is shown that the general structure of such an oscillator includes four distinct regions; the outer two regions correspond to very small or very large amplitude of the external force and match the corresponding regions in soft oscillators (invertible degree one and degree zero circle maps, respectively). There are two new regions for intermediate amplitude of the forcing. Region 3 corresponds to moderate high forcing, and is intrinsic to hard oscillators; it is characterized by discontinuous circle maps with a flat segment. Region 2 (low moderate forcing) has a certain resemblance to a similar region in soft oscillators (noninvertible degree one circle maps); however, the limit set of the dynamics in this region is not a circle, but a branched manifold, obtained as the tangent union of a circle and an interval; the topological structure of this object is generated by the finite size of the repelling set, and is therefore also intrinsic to hard oscillators.

I. INTRODUCTION

The dynamics of relaxation oscillators subject to a periodic excitation has been extensively studied in the past few years. In the limit of strong relaxation, the dynamics of the system in large sections of parameter space is adequately depicted by maps of the circle. The structure of the stability portrait in these regions of parameter space is well understood and is known to be stable under reasonable perturbations.

Changes in the topology of the oscillator will, however, affect the structure of the stability portrait. The basic oscillator type that has been studied is the soft oscillator; its topological structure consists of a stable limit cycle with an unstable fixed point inside. A different type of oscillator is the hard oscillator, which again has a stable limit cycle, but it has a stable fixed point inside, and an unstable limit cycle separating both attractors. See Fig. 1. They are called hard because it is necessary to excite them to obtain oscillations when the initial condition lies near the fixed point, while the soft oscillators will always oscillate spontaneously unless placed exactly on the fixed point. Nature and technology abound in examples of both; see Ref. 11 and references therein.

We review in Sec. II the techniques needed to deal with soft oscillators. In Sec. III, we present a hard oscillator with closed form solutions for the homogeneous time evolution. In Sec. IV, we study the behavior of this model subject to periodic kicks and show that the stability portrait suffers generic alterations from the “classical” structure associated to soft oscillators. We study the system in the strong relaxation limit. There are four distinct regions, which we number 1 to 4 in order of increasing strength of the forcing amplitude. Regions 1 and 4 (circle diffeomorphisms and degree zero maps) are exactly equivalent to those found in soft oscillators; since they are well understood we will not dwell on them here. In region 3 the amplitude of the forcing is large enough to send the oscillation into the domain of attraction of the stable fixed point; the phase of the oscillation is annihilated and the overall dynamics can be described by a discontinuous map of the circle with a flat segment. We consider this in Sec. V. Region 2 corresponds to moderate low amplitudes, where the oscillation, without being able to go inside the unstable limit cycle, can nevertheless interact with it. There is a similar region in soft oscillators; however, the finite size of

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the unstable limit cycle will drastically change the structure of this region. We analyze this region, in its strong relaxation limit, in Sec. V; we show that the most dramatic change is that the attracting set is not a circle anymore, but a one-dimensional set with a more complex topological structure: a branched manifold made as the tangent union of a circle and a segment.

II. SOFT OSCILLATORS

We will review in this section known features of soft oscillators which we will need in the latter sections.

The structure of periodically kicked oscillators in general is the following: a kick is applied, therefore mapping the state of the oscillator to some other state. Between kicks, homogeneous time evolution of the system takes place.

Arbitrarily complicated dynamics can be obtained by complex kicking structures. We will only consider in this paper a particularly simple form of kicking, a translation in phase space by an amount \( F \), independent of the state of the system (i.e., n-parametric). See Fig. 2. For this form of kicking, all of the interesting geometric properties have to do with the homogeneous time evolution, and we will be forced to make constant references to the invariant sets of the unforced oscillator. Henceforth, the terms “limit cycle” and “fixed point” will refer exclusively to the invariant sets of the unforced oscillator, even when in the context of forcing.

Let

\[
\dot{x} = f(x,y), \\
\dot{y} = g(x,y)
\]

be an unforced soft oscillator. Then it is always possible to find a global coordinate transformation \((x,y) \rightarrow (A,\phi)\) such that

\[
\dot{A} = -\mu A, \quad \dot{\phi} = 2\pi,
\]

where \( A \) can be interpreted as a “distance to the limit cycle,” defined as zero on the limit cycle, negative inside and positive outside, \(-\infty\) on the fixed point. \( \phi \) is the “phase” of the oscillation. The procedure for achieving this coordinate transformation is as follows. On the limit cycle, \( \phi \) can be defined in the obvious fashion, as an elapsed time from a designated origin of phases divided by the period; it can then be extended to all of phase space minus the fixed point, by constructing the stable manifold foliation of the limit cycle and defining \( \phi \) to be constant on its leaves, which are called isochron curves. \( A \) can similarly be defined on a small annular neighborhood of the limit cycle, and extended by reversed time evolution. That this map is global is a direct consequence of the basin of attraction of the limit cycle being all of phase space minus the fixed point, so that inverse images of a neighborhood of the limit cycle cover phase space. We can always define \( A \) in such a way that \( |A| \) coincides with ordinary distance along the isochron curves for very small \( A \), and then \( \mu \) will automatically be a constant when the divergence of (1) on the limit cycle is bounded. All of the peculiar properties of each oscillator are hidden by the coordinate transform.

This transformation is global, one to one, and diffeomorphic for all \( \phi \) and all finite \( A \). It cannot, however, transform diffeomorphically any oscillator into any other one, since the eigenvalues of the fixed point must be preserved by a diffeomorphism. This will have interesting consequences later.

We will subject this oscillator to a forcing of the form

\[
\dot{x} = f(x,y), \\
\dot{y} = g(x,y) + F \sum \delta(t - n\tau).
\]

We could expect that in the limit of large \( \mu \), for suitable forcing amplitudes and frequencies, the states of the oscillator just before the kicks will lie very near to the limit cycle, since the oscillator contracts phase space at a rate which in the vicinity of the limit cycle is of the order of \( \exp(-\mu \tau) \). The dynamics of the system then collapses to the dynamics of the limit cycle, and could therefore be described by maps of the circle. We will now make this argument a little more precise.
Let \( M(A,\phi) = (x, y) \). Let \( F_c \) be the amplitude of the kicks and \( \tau \) their spacing. Then, calling \((A_n, \phi_n)\) the coordinates just before the kick numbered \( n \), and \((A'_n, \phi'_n)\) the coordinates just after the kick,

\[
(A'_{n+1}, \phi'_{n+1}) = M^{-1}(M(A_n, \phi_n) + (0, F_c)),
\]

where the first equation is the expression of the kick, and the second one is obtained through integration of Eq. (1).

It is clear from the geometry that if \( \mu \) is large enough, then all \( A_n \) will be near 0; we can then expand the above equation around \( A_n = 0 \) and neglect terms \( O(\mu) \). We therefore obtain (calling \( M^{-1} \) the \( \phi \) component of \( M^{-1} \))

\[
\phi_{n+1} = M^{-1}(M(0, \phi_n) + (0, F_c)) + 2\pi \tau
\]

(4)

which is a map of the circle. The conditions under which this is valid are

\[
\max_{\phi} \frac{\partial \phi_{n+1}}{\partial \phi} \big|_{A_n=0} < 1, \quad \max_{\phi} \frac{\partial \phi_{n+1}}{\partial A_n} \big|_{A_n=0} < 1.
\]

(5)

These conditions define the strong relaxation regime as a region in parameter space. In the \( F_c \) vs \( \tau \) plane, the first inequality has roughly the form \( F_c \exp(-\mu \tau) < 1 \) [as can be seen from Eq. (3)]; the second inequality is well satisfied anywhere in the \( \mu \to \infty \) limit, except when \( F_c \) is such that the image of the limit cycle after the kick goes through the fixed point. This is the line usually called the critical line; we will call this critical amplitude of the forcing \( F_c \).

The critical line is always outside the realm of strong relaxation, at least by this simple argument. The reason is the following: no matter how strong relaxation is, if a point in the limit cycle gets kicked precisely to the fixed point, it will never return to the limit cycle by itself. [Strictly speaking, the first inequality in Eq. (5) also excludes the critical line, but the second inequality is still more stringent due to faster scaling.]

The phase portrait of kicked soft oscillators is typically divided by the critical line into two regions. For \( F_c < F_c \), (hereafter called region 1, also called subcritical in the literature) the image of the limit cycle under the kick still contains the stable fixed point in its interior, and therefore intersects all isochron lines. For \( F_c > F_c \) (hereafter region 4, also called supracritical in the literature), the image of the limit cycle does not contain the fixed point, and therefore intersects only a subset of the isochron lines. See Fig. 3. In region 1, as the phase of a point in the limit cycle is varied through one turn, its image under the kick intersects every isochron precisely once; therefore winds around the fixed point also once. The circle map (4) therefore has topological degree 1 (Ref. 25) and is a diffeomorphism; in particular, it is invertible. This region, which is monostable (in the sense of only one attracting set), contains synchronization regions (Arnold tongues) and quasiperiodic responses. It has been explored in a great deal of detail and is essentially, completely understood.

In region 4, the image of the limit cycle under the kick does not wind around the fixed point, only a subset of all phases is explored, and (4) has topological degree 0; it is therefore equivalent to a bimodal map of the interval. This region contains bistable responses (having precisely two attracting sets), so that slow variations of parameters can give rise to hysteresis effects, and cannot contain stable quasiperiodicity, but may contain synchronization regions and chaotic bands. It is much more complex than region 1, but has nevertheless been studied extensively and is also well understood.\(^{1,8,10}\)

Please bear in mind that in the circle map literature, subcritical refers to a map of degree one which is invertible, and therefore the critical line is the curve in parameter space where the map acquires an inflection point.\(^7\) Although degree zero implies noninvertibility, the converse is not true, and there is nothing in the structure of these oscillators that prevents them from losing invertibility before changing topological degree.\(^{13,4,16}\) We will now show that this is actually the typical behavior.\(^9\)

Consider an oscillator whose isochron curves gracefully terminate at the fixed point, in the manner shown in Fig. 4. This is the oscillator type that has been most extensively studied because of its simplicity.\(^9\) Since the rate of change of phase is fixed by (1), the linearized angular frequency around the fixed point (the imaginary part of the eigenvalue) must be equal to the frequency of the limit cycle. This is a nongeneric situation, the generic situation being that they are different. A generic oscillator cannot be mapped to this particular type because the eigenvalues of the fixed point and the frequency of the limit cycle cannot change under a diffeomorphism. Therefore, for a generic soft oscillator, the isochron curves cannot terminate gracefully on the fixed point; they will have to wind around the fixed point, precessing at the difference in frequency. Their generic shape near the fixed point will be a sheared logarithmic spiral, getting nearer to the fixed point by a factor \((\omega_{LC}/\omega_{FP} - 1)/\mu\) with each turn around it, where \( \mu \) is the relaxation rate of the fixed point, \( \omega_{FP} \) its linearized frequency, and \( \omega_{LC} \) the frequency of the limit cycle.

This winding around has drastic consequences in that
band around the critical line which lies outside the strong relaxation regime. If \( F_a \) is smaller than \( F_c \), but very near it, the image of the limit cycle under the kick will intersect a given isochron curve several times, as shown in Fig. 5. This means that the map is no longer one to one, and that invertibility has been lost. There is therefore, a critical line, in the standard sense of circle maps, just below the change in degree. Between these two lines, we have region 2 (of soft oscillators). Its height in parameter space may decay to zero with increasing relaxation, but it can be nevertheless expected to possess a vastly complex topological structure. This region has not been studied yet in as much detail as regions 1 and 4.

Attractors which have been reconstructed from the numerical evolution of the full equations of motion in this region (as opposed to the circle map which assumes a collapse onto the limit cycle) show clear one dimensionality and collapse to a topological circle. Small fluctuations around the lower part of the limit cycle become important because they can be amplified when kicked near the fixed point, where the gradient of phase has a singularity. Because of this singularity the naive conditions (5) fail, but

the system is one dimensional in nature and well described by circle maps. These maps (representing the dynamics collapsed onto the attractor) can also be reconstructed from numeric data. Due to the small perturbations being important, they differ from the maps calculated from (4), which are discontinuous on the critical line; this difference is directly due to the fact that the attracting set differs from the limit cycle and the singularity amplifies this difference. They are degree one maps which have lost invertibility. Their most noticeable feature is that they depend on \( \tau \) in a nontrivial manner. We will not address this issue further.

Therefore, except the provision that maps on the critical line cannot be calculated from the naive expansion (4), we can safely say that region 2 can be included in the strong relaxation regime, at least in terms of the attracting set being, topologically, a circle. The dynamics is that of noninvertible circle maps of degree one.

III. A HARD OSCILLATOR WITH CLOSED FORM SOLUTION

When we try to apply the arguments of the previous section to hard oscillators, we immediately see a breakdown. The basic property of soft oscillators which allowed us to obtain a generic global coordinate transformation is the basin of attraction of the limit cycle being dense. However, for hard oscillators the basin of attraction of the stable limit cycle does not contain the unstable limit cycle and its interior, and a coordinate transformation similar to the one defined above is not global. A simple argument shows the precise nature of the breakdown. Let \( T \) stand for the map of phase space obtained by evolving the oscillator for a time equal to the period of the outer limit cycle. The limit cycle is then a set of indifferently stable fixed points, attractive in the direction perpendicular to itself. Call \( S \) the stable manifold of a point on the limit cycle; \( S \) is an isochron, a smooth connected curve which is left invariant by \( T \) and \( T^{-1} \), and such that \( T \) is a contraction in the vicinity of the limit cycle. Assume that the isochron \( S \) intersects the unstable limit cycle transversely. Therefore, all other isochron curves intersect it transversely (by invariance of the intersection, see Ref. 23). Furthermore this means that the intersection point of \( S \) with the unstable limit cycle is a fixed point of \( T \), so that, by an argument similar to the one in Sec. II, the frequency of both limit cycles is the same. Therefore, in the generic case in which they have different frequencies, \( S \) has to wind densely around the unstable limit cycle. Since, as it was the case before with soft oscillators, this is not a resonance problem but a problem of matching angular frequencies, transverse intersections are achieved only for equal frequency cycles, not for rational or integer quotients. This can be seen also from the fact that the invariant manifolds evolve continuously as a function of the parameters.

We will study one particular oscillator, whose homogeneous equations can be solved exactly, so that a closed form stroboscopic map can be obtained.

Our model is defined by the following ode:
\[
\dot{r} = \mu f(r), \quad \dot{\theta} = \frac{4\pi}{r+1},
\]
where
\[
f(r) = \begin{cases}
-\frac{r^2}{r_c} & (r < r_c) \\
\frac{(r - r_c)^2}{(1 - r_c)} & (r_c < r < 1) \\
1 - r & (r > 1)
\end{cases}
\]

So \( f(r) \) has three zeros, at 0, \( r_c \), and 1, which will be the fixed point, unstable limit cycle and stable limit cycle, respectively. Between these zeros, \( f(r) \) has the shape of parabolic arcs with derivatives \( \pm 1 \) on the zeros. Beyond \( r = 1 \), \( f(r) \) is a line with slope \(-1\). \( \theta \) has been defined so that the frequency of the unstable limit cycle is greater than that of the stable limit cycle, whose period is 1. Notice that this ode is \( C^1 \), piecewise \( C^\infty \). It lacks smoothness only at the zeros of \( f \); therefore, since these points cannot be reached in finite time by the flow, time evolution is \( C^\infty \) everywhere.

We can solve this ode in closed form, to obtain
\[
r(t) = \frac{r_0 e^{-\mu t} + r_0}{(r - r_0)e^{\mu t} + r_0},
\]
\[
\theta(t) = \theta_0 + \frac{4\pi t}{1 + r_c} + \frac{4\pi r_c}{\mu (1 + r_c)} \ln \left( \frac{1 + 1/r(t)}{1 + 1/r_0} \right) \quad (r_0 < r_c),
\]
\[
r(t) = r_0 + \frac{(1 - r_c)(r_0 - r)}{(1 - r_0)e^{\mu t} + (r_0 - r_c)},
\]
\[
\theta(t) = \theta_0 + 2\pi t - \frac{2\pi}{1 + r_c} \ln \left( \frac{r_0 - r_c}{1 + r_0} \frac{1 + r_c}{1 + r(t) - r_c} \right),
\]
\[
r(t) = (r_0 - 1)e^{\mu t} + 1
\]
\[
\theta(t) = \theta_0 + 2\pi t + \frac{2\pi}{\mu} \ln \left( \frac{1 + r(t)}{1 + r_0} \right) < r_0.
\]

We will force this oscillator by kicking the \( y \) coordinate up, by an amount \( F_0 \) every \( \tau \) units of time. The stroboscopic map \( M(r, \theta) \) is then the composition of two maps, \( M = M_2 \circ M_1 \), with \( M_1 \) reflecting the effect of the kick and depending only on \( F_0 \) its amplitude; and \( M_2 \) being homogeneous time evolution and depending only on \( \tau \), the spacing between kicks. Both can be written in closed form.

\( M_1(r, \theta) \) is the polar coordinate form of the map \((x, y) = (x, y + F_0) \cdot (r, \theta) = M_2(r_0, \theta_0) \) is given by Eq. (8) with \( t = \tau \).

**IV. ALTERATIONS TO THE STABILITY PORTRAIT**

In hard oscillators, for sufficiently small or sufficiently large \( F_0 \), the image of the stable limit cycle after a kick will not intersect the unstable limit cycle. Dynamics will therefore remain in the region where the isochron network is a well-defined coordinate system, when relaxation is strong. For small \( F_0 \), in our model \( F_0 < 1 - r_c \), this image will intersect all isochron curves and the dynamics will be described by an invertible map of the circle with topological degree one. The same features as in soft oscillators will be observed: unistability and Arnold tongues following a Farey tree numerology separated by quasiperiodicity. For large \( F_0 \) \((> 1 - r_c)\) the image of the stable limit cycle after a kick will clear the repelling region, and intersect a subset of all isochron curves. This region will be characterized by a map of the circle with topological degree zero. Both these regions are expected to be structurally identical to those in soft oscillators.

When \( F_0 \) is such that the image of the stable limit cycle under \( M_1 \) intersects the inner limit cycle, thereby mapping the stable limit cycle into a region of phase space where the phases have not been defined, we find another region where different behavior occurs. We will call this region 3 and will study it in some detail in Sec. V.

By the same argument as in Sec. II, due to the dense winding of the isochron curves around the unstable limit cycle, the circle map for these oscillators in region 1 will lose monotonicity before region 3 is encountered. We will call region 2 \( (\) of hard oscillators \( ) \) the region between the line on which monotonicity is lost and region 3. We will make a distinction between this region 2 and that for soft oscillators \( (\) where it corresponds to noninvertible degree one maps \( ) \), because of important qualitative differences which will be exposed in Sec. VI.

In our model, due to the map being a composition of two maps with each of them depending on only one of the external parameters, all boundaries between regions are straight lines, horizontal in the \( F_0 \) vs \( \tau \) plane. Region 1 lies in \( F_0 < 1 - r_c - h(r_0, \mu) \), region 2 in \( 1 - r_c - h < F_0 < 1 - r_c \), region 3 in \( 1 - r_c < F_0 < 1 + r_c \), and region 4 above, with \( h \) being an exponentially decreasing function of \( \mu \). Figure 6 shows a global stability diagram for our model in high relaxation. Region 2 is too thin to be seen at this resolution. We summarize the overall situation in Fig. 7, where we show the relationship between the geometry of the oscillator, the regions of the stability portrait and the maps describing the dynamics in each.

In contrast with the situation in soft oscillators, where the attracting set is complex but the repelling set is virtually trivial (being an unstable fixed point contained inside the limit cycle), in hard oscillators we observe an immensely rich repelling structure. To see this, imagine time going backwards in this oscillator \((\) or \( \mu \) and \( F_0 \) being negative \( ) \). Then the time reversed oscillator is a soft oscillator \((\) with the stable limit cycle being the inner one \() \) enclosed in an unstable limit cycle. Therefore, for suitable values of the parameters, the attracting set of the time-reversed oscillator \((\) which is part of the repelling set of the time-forward one \() \) undergoes all of the quasiperiodic-chaos transitions seen in soft oscillators. In some regions of parameter space this rich repelling set can interact with the attracting set in complex ways; for instance, when \( 1/3 < r_c < 1/2 \) and \( F_0 \) is smaller than \( r_c \) but near it, we may have a chaotic attractor and a chaotic repeller which intersect. This, however, is outside the scope of this paper and will be considered elsewhere.
V. REGION 3: DISCONTINUOUS CIRCLE MAPS

In this region, new synchronization regions are born. These have the property that one of the pulses falls inside the unstable limit cycle. They are strongly stable because of this, since the fixed point is attractive in all directions, unlike the limit cycles which are marginal along themselves. Because of this the limit $\mu \to \infty$ is perfectly well behaved, and the infinite $\mu$ regime is qualitatively the same as finite, but large, $\mu$. This section refers to this high relaxation regime indistinctly, even though the calculations are actually made for infinite $\mu$.

The flow cannot support two different phase-annihilated orbits simultaneously, so that the region is monostable. By the same token, one of these orbits cannot have two cycle elements inside limit cycle.

The limit set on which dynamics takes place in the infinite $\mu$ regime consists of the stable limit cycle plus the fixed point. In this regime, any point in phase space is carried by the homogeneous time evolution to the attracting invariant sets in zero time; after that, it just precesses in phase. So $M_3(r,\phi) = (1, \phi + \tau)$ if $r > r_c$, and $(0,0)$ otherwise. Any iteration falling inside the unstable limit cycle will be “phase annihilated” to the fixed point. We can then write the map in the infinite relaxation limit in the following way [using complex number notation $(x,y) \to z=x+iy$]

$$z_{n+1} = \begin{cases} \frac{z_n+iF_c}{|z_n+iF_c|} e^{i\pi r}, & |z_n+iF_c| > r_c, \\ 0, & |z_n+iF_c| < r_c. \end{cases}$$

We can construct a map of the circle with this time evolution in a simple manner. There is a region $\Omega$ of the outer limit cycle which gets mapped to the fixed point. The fixed point in turn gets mapped to $\theta = +\pi/2+\tau$. Therefore we can construct a map of the circle only, where $\Omega$ is mapped directly to $+\pi/2+\tau$; but we have to remember that, for orbits with elements in $\Omega$, the “real” period in the original system is the period in the map plus one. Setting $s_n = \exp(i\theta_n)$, we have, in complex notation, a map from $\theta_n$ to $\theta_{n+1}$:

$$s_{n+1} = \begin{cases} \frac{s_n+iF_c}{|s_n+iF_c|} e^{i\pi r}, & |s_n+iF_c| > r_c, \\ \exp\left(i\frac{\pi}{2}+2\pi i\tau\right), & |s_n+iF_c| < r_c. \end{cases}$$
FIG. 6. The stability diagram for our model, global view. Vertical axis is $F_\phi$, varying between 0.8 and 1.2. Horizontal axis is $\tau_c$, varying between 0.1 and 0.5. $r_c = 0.05$, $\mu = 40$. Color code for all color plates is the following. A given color represents a period; aperiodic responses (such as chaos or quasiperiodicity) and periods beyond the resolution of the plot (64) are shown in black. Periods which are double another period are shown in the same color hue but a lighter shade. Period 1 is blue (with periods 2, 4, etc., being successively lighter shades of blue); period 3 is green, 5 red, 7 yellow et cetera. The plots were made by scanning the parameter space from left to right, which creates some artifacts in bistable regions.
We can see a plot of this map on Fig. 8. Notice the flat section mapping $\Omega$ to $\pi/2 + \tau$; this discontinuity makes the topological degree of the map undefined. Any phase falling into this segment is “annihilated” to a constant value.

When relaxation is high but finite, the orbits of this map have a very clear signature when only one of the variables ($x$, for instance) is plotted as a function of time, as would be observed on an oscilloscope or on Fig. 9. This signature consists of snaky segments (bracketed precisely by two pulses) showing a decaying oscillation of smaller amplitude than the main oscillations. The other clear signature is that the measure of parameter values corresponding to mode locking is complete over a finite range of $F_e$.

The behavior in this region bears strong relationship to the phase resetting oscillator described in Ref. 5 in that, unlike the region 1, tongues touch each other directly and there exist only a finite number of tongues at any given $F_e > 1 - r_e$. There are some differences between the hard oscillator and Allen's though: in our case, for low $F_e$, the preimages of $\Omega$ do not cover the circle densely, so that the measure of parameter values corresponding to mode locking may not be complete, and stable periodic orbits exist which do not go through the flat segment (do not fall inside the unstable limit cycle). Furthermore these can period double. Figure 10 shows a detail of the stability portrait of (12), and Fig. 11 a bifurcation diagram for this model.

VI. REGION 2: BRANCHED MANIFOLDS

This is the most interesting region of this model. In soft oscillators it has been scantily studied; but numerical simulations do not show any big surprises. In hard oscillators interesting things happen. One would naively expect from Fig. 5 that one could see an arbitrarily large number of intersections between the image of the limit cycle and the isochron network. This, however, does not happen, because the attracting set of the dynamics is no longer the limit cycle, even in the limit of infinitely high relaxation. But the hard oscillator brings further problems: the topology of the attracting set changes, and it is no longer a circle.
but a branched manifold. Thereby, we cannot refer to this region as “noninvertible degree one,” as we did for soft oscillators.

At high relaxation it is extremely thin in parameter space and might be missed by purely numerical studies, but its thinness is irrelevant for a study of the topology of the phase portrait and the onset of transitions. In region 1, which lies below this region in $F_r$, the oscillator does not support the foldings necessary to construct disorderly behavior. The basic mechanism through which the foldings proceed in this region is localized near the unstable limit cycle and acts as follows.

Since the frequency of the inner limit cycle is higher than that of the outer limit cycle, initial conditions near the former, when allowed to evolve through a time $\tau$, will have shifted their phase by a larger amount than initial conditions further from the unstable limit cycle. This can be seen clearly from (8). This extra shift in phase is logarithmic in the distance to the unstable limit cycle. In the forced behavior, this means that if $F_r=1-r_c-\epsilon$, with $\epsilon$ a small positive number, $M_1$ maps the outer limit cycle very near to the inner limit cycle, but without touching it. $M_2$ makes the points which are not mapped near the inner limit cycle shift in phase by approximately $2\pi \tau$. But the point $(x=0,y=-1)$ and its immediate neighbors will acquire an extra shift in phase proportional to $\log(\epsilon)$. Therefore, if $\epsilon$ is small enough, the new phase after one cycle of the excitation, as a function of the phase prior to the excitation, will in the neighborhood of $\theta=-\pi/2$ first rise, as it approaches $-\pi/2$ on the left, then fall. It is this fall, the loss of monotonicity, which provides the folding of phase space essential to chaotic behavior.
FIG. 10. Stability diagram of region 3. $\mu = 40, \tau_r = 0.3$; vertical axis is $F_r$, varying between 0.7 and 1.3. Horizontal axis is $\tau$, varying between 0.1 and 0.5.
It is this very same mechanism that makes the limit set on which dynamics takes place differ from the limit cycle. Points which are mapped near the unstable limit cycle tend to get stuck to it, the escape time being also logarithmic in the distance. Precisely on the critical line $F_e=1-r_c$, the point $(x=0, y=-1)$ gets mapped to the unstable limit cycle and will not relax to the outer limit cycle. The limit cycle is mapped, then, to a concave shape, and is not an invariant set. Please see Fig. 12, where a sketch of the time evolution of a circular segment barely touching the unstable limit cycle is shown. We can see that this mechanism concentrates simultaneously all the stretching and folding that occurs in this region. Since the conditions determining the boundary of the region are precisely those which determine the deformation of the limit set, the alteration of the limit set occurs throughout the region.

The angular width of the region where this effect takes place is exponentially small in $\mu$, as can be seen directly from (8). Therefore, as relaxation is increased, the angular width of the concavity diminishes. Any chaotic orbit is obliged to have measure on this concavity, since it is its preimage which supports foldings. In Fig. 13, we show the limit set of the dynamics, on the left edge of the period 5 window, for different $\mu$. The limit set has roughly the shape of the PacMan icon; for low $\mu$ its mouth is wide open, and as $\mu$ is increased its mouth closer exponentially fast. In the limit $\mu \to \infty$ we obtain a branched manifold. Notice that the branch joins the circle tangentially, so that the tangent space to this manifold is well defined everywhere, including the joint. The plots have been made precisely on the top of the region because that is the only $F_e$ that will remain inside the region for all values of $\mu$, but the same type of alteration will occur for lower $F_e$'s still contained in the region. In fact, an alternative definition of region 2 can be made through the deformation of the limit set.

The bifurcation diagrams themselves on the upper boundary of the region show interesting and unusual features. In Fig. 14 we see the projection on the attracting set onto $\theta$, as a function of $\tau$. A blowup of the period 3 window can be seen in Fig. 14. It shows a most unusual feature for a forced oscillator: The window is born on the right out of quasiperiodicity by an intermittent transition, but it dies on the left by period doubling. While this is the standard setup for a map of the interval, in circle maps the outer borders of Arnold tongues are always intermittent, since they have to be so in region 1 and they preserve this property by continuity. The change in sign of the stability parameter necessary for period doubling occurs towards the center of the tongue, so that the tongue has a somewhat symmetric structure. See Fig. 15 for the bifurcation diagram of the period 3 window in a soft oscillator just above the lower boundary of region 4.

It would seem that the above argument by which one shows the "symmetry" of Arnold tongues in soft oscillators should also hold here. A stability portrait of the critical band around the period 3 tongue shows what happens (Fig. 16). Period doublings effectively occur on the centroid of the tongue. But the left half of the (now forked) tongue is extinguished before it reaches the upper bound-
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FIG. 14. Bifurcation diagram of the period 3 window on the critical line. Notice the period doublings on the left side of all periodic windows.

FIG. 15. Period 3 window for a soft oscillator on the critical line. Notice the absence of period doublings throughout.

FIG. 16. Stability portrait of the 1:3 tongue in region 2, $\mu=40$, $r=0.3$; vertical axis: $0.6995 < F < 0.7$, horizontal axis $0.15 < \tau < 0.4$. Notice how the tongues period-double and split quite asymmetrically. The left branch never reaches the "critical line" separating region 2 from region 3 (the top boundary of the plot), so that on the critical line (depicted in Fig. 14) all periodic windows have a period doubling cascade to their left side.
ary of the region. The reason why the symmetry of the tongue is so badly broken stems from the folding mechanism (which splits the tongue in the first place) being badly asymmetrical itself.

Qualitatively or topologically the shape of the tongues in region 2 is the same for soft and hard oscillators; quantitatively there is a vast difference. In soft oscillators the limit set is "near" the limit cycle, in the sense that it deviates from it by small perturbations exponentially small in \( \mu \). In the hard oscillator case, the limit set is not near the limit cycle, neither does it tend to be it. The set has approximately the topology of a circle for all finite \( \mu \); it actually has a Cantor set structure, due to the images of the mouth of the "PacMan," but these images get flattened very fast with a scaling factor which is exponential in \( \mu \). [Only on Fig. 13(a) can the first of these images be observed just at the figure's resolution.] As \( \mu \to \infty \), this structure becomes less essential to the dynamics and is lost; we regain a smooth limit set.

The lesson to be learned from this system has to do with the mechanics of dimensional collapse due to dissipation. While it is clear that under infinitely strong dissipation dynamical systems will collapse onto the lowest dimensional manifold that can support their dynamics, it is not the case that these manifolds will be simple topologically. In our system the interaction with a repeller of finite extent leads naturally to the branching of the limit set, since the flow only supports foldings on that branch. It is to be expected that higher dimensional systems having more complicated dynamics will lead to limiting sets with weirder topologies. An understanding of these topologies and the smooth dynamics they can support will clarify and help classify the allowed dynamics of these systems when relaxation is not infinite and a substantial use of the full dimensionality of phase space is being made, much in the same way the understanding of the allowed dynamics for maps of the interval and of the circle (the simplest one dimensional examples) has clarified and structured our understanding of soft oscillators (for instance) even when dissipation is not strong.

VII. CONCLUSIONS, CAVEATS, AND PERSPECTIVES

We have presented a simple model of a kicked hard oscillator having a closed form stroboscopic map. We have shown that the finite extension of the repelling set, and the presence of a second attracting set lead to alterations in the structure of the stability portrait in parameter space. We have analyzed the structure of the limiting sets of the dynamics as the relaxation increases to infinity, and have shown that in a narrow area in parameter space the limiting set is not the stable limit cycle, as would be expected from naive arguments, but a set with a slightly more complex topology, possessing a branch attached tangentially to the limit cycle. Outside this narrow area the limit set is the limit cycle (region 1), a subset of the limit cycle (region 4), or the union of the limit cycle and the stable fixed point (region 3).

The model we have presented cannot, however, aspire to become a "universal" model of hard oscillators. We have already shown explicitly that there is no mapping transforming these oscillators to a standard form like (1). This does not mean that there is not a more complex way of achieving this, but it does mean we should be careful about claiming genericity. The truly generic features of these oscillators are the presence of regions 2 and 3. In particular, one nongeneric aspect of our model is that relaxation rates around the limit cycles and the fixed points are all the same, namely, \( \mu \). A model of hard oscillators attempting to reach universality will need at least a six-dimensional parameter space, since the frequency of the unstable limit cycle, the imaginary component of the flow gradient eigenvalue at the fixed point, the relaxation rates around both limit cycles and the fixed point, plus the radius of the unstable limit cycle should be independently regulated. (The frequency and radius of the outer limit cycle can be absorbed in the units of time and space.)

We have exposed the phenomenological aspects of our model, as well as those aspects related to geometric theory of ODEs. We will expose the aspects related to global stability analysis (like global layout of the new regions) and ergodic theory (like the dynamics supported on them) in a sequel to this paper.

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24 Strictly speaking one must allow for an odd number of intersections of the limit cycle with the vertical downwards ray starting at the fixed point, therefore defining an odd number of critical lines and a number of sub- and supracritical regions; such oscillators, however, cannot be realized as second-order odes in one degree of freedom, and are not small perturbations thereof, therefore excluding most electronic and electromechanical oscillators.
25 The topological degree of a map of the circle is defined as
\[ \deg(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{df}{d\theta} d\theta \]
and represents the number of times the function winds around the circle as the argument winds once.
26 In addition, if \( F < r_c \), we will observe an extra stable limit set, since an initial condition inside the unstable limit cycle will be trapped inside if \( F \) is small enough and \( \mu \) and \( \tau \) are large enough. There is therefore an extra attracting set for some parameter values.