Correlations in cellular patterns

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ABSTRACT

Cellular domain patterns arise from a competition between a minimization process (such as surface tension) and geometric constraints; from this competition a topological structure emerges which dominates the patterns' geometry and dynamics. The best-studied cellular system is the two-dimensional soap froth, but, even though it is conceptually and formally the simplest member of the family, its long time and long distance properties are far from elucidated.

We study the spatial correlations induced into the topological structure of cellular patterns by both the dynamics and the embedding process that defines the geometry of the patterns and show that these correlations strongly deviate from those of a freely fluctuating topology.

§1. INTRODUCTION
1.1. Motivations

The most important open issue in the theory of soap froth and cellular patterns has been, for four decades now, that of the scaling state. Two-dimensional soap froth coarsens continuously in time; Smith (1952) conjectured that this coarsening is such that the system approaches statistically self-similar evolution, meaning that, after sufficient time has elapsed and transients have died away, the only relevant change in the system is its average length scale; if we scale the froth to unit average length scale, the statistics of the froth will remain stationary in time. Smith's conjecture rested on apparently shaky experimental grounds and appeared to have been disproven by Aboav (1970). Later, Glazier and Stavans carried on some more careful experiments which strongly indicated the conjecture to be true and established that Aboav's samples had not been sufficiently equilibrated (Glazier, Gross and Stavans 1987, Glazier and Stavans 1989, Stavans and Glazier 1989). This precipitated a cascade of studies which confirmed Smith's scaling conjecture from several different viewpoints: mean-field theories (Fradkov, Shvindlerman and Udler 1987, Marder 1987, Mullins 1988), detailed simulations (Weaire and Lei 1990), topological models (Beenaker 1986, Fradkov et al. 1987, Fradkov and Udler 1990), $\mathcal{Q}=\infty$ Potts models (Glazier 1990, Glazier, Anderson and Grest 1990) and finally the definitive experiments of Stavans (1990).

Even after this cascade of work, the scaling conjecture remains at present unproven. The technical difficulty of the ideal soap froth model, as defined by Smith and von Neumann, is still above our craft. The problem can be stated in terms of elementary trigonometry and yet seems to require introduction of rather heavy-handed mathematical machinery. My aim in this paper will be to lay out some formal grounds which I believe will be useful in the study of this issue.

The structure of this paper is as follows. The remainder of this section will be devoted to classical froth theory. In §2 the topology and related issues are described. In
§ 3, I shall introduce some issues of global structure of cellular patterns, which seem to have been long neglected; in particular, I shall define the ‘intrinsic dimension’ of a cellular system. § 4 is devoted to the interplay between ‘extrinsic’ and ‘intrinsic’ properties and, in particular, to the correlations which arise from the geometry used to actually draw the abstract topologies. Finally, in § 5, I shall attempt to conclude something meaningful from the preceding tangle. A review of the (rather abstract) techniques of conformal invariance and the Teichmüller spaces is given in Appendix A, and a glossary of mathematical jargon is provided in Appendix B.

The main results in this paper can be summarized as follows. In § 2, I show that the correlations existing between topological charges for any topology are essential to the global structure of the systems; in particular, mean-field theories based upon joint probability distributions of charge and areas are not an approximation to the real dynamics in a formal sense, and they are logically inconsistent. This should give the reader an idea of why the scaling conjecture is hard to prove. In § 3, I compute intrinsic dimensions for some static models, in particular the random Voronoi model. In § 4.3, I show that, for a cellular pattern, local averages of topological charge are dependent upon local variations in average length scale. I use this fact to show, in § 4.5, the main result, that soap froth evolution tends to equilibrate differences in length scale with a screening length of the order of one bubble, this screening length being now formally the same as that for topological charges.

1.2. The ideal soap froth

Two-dimensional cellular patterns are named after their resemblance to living tissue. It was already noted by the last century that their emergence was not at all circumscribed to the histological domain but, quite the contrary, these patterns appear almost unchanged in many different systems, across several different disciplines, from crack patterns in lava slopes to territorial patterns in ecology (Weaire and Rivier 1984, Glazier 1989, Glazier and Weaire 1992). Thompson (1917) guessed quite rightly that their ubiquity is not the result of detailed complex system-dependent interactions; it is the outcome of a competition between a minimization process and global constraints of a geometric nature. Unfortunately he did not proceed to obtain a general theory of froths but chose to apply some general ideas back to the realm of biology. The modern history of the subject begins when Smith (1952), at a metallurgy conference in Cleveland, proposed the study of two-dimensional soap froth as a means for understanding a problem of vastly larger practical importance: that of evolution of grains inside a metal. The grain structure of metals underlies most properties of metals as we known them, from elasto-plastic to electric. John von Neumann was at Smith’s seminar; the enfant terrible of Princeton’s mathematicians was researching materials for magnetic core memory for his computer. He immediately derived the equations of motion of soap froth (von Neumann 1952), which today bear his name.

Ever since, the study of two-dimensional cellular patterns proceeded, mostly along three distinct but sister branches (Weaire and Rivier 1984, Glazier 1989): soap froth à la Smith–von Neumann (Smith 1954, 1964, Weaire and Kermode 1983, 1984); motion by curvature (‘boundary-dominated grain growth’) (Harker and Parker 1945, Smith 1948, Fradkov et al. 1987, Fradkov and Udler 1990, Mullins 1988, Ralph 1990); motion by tangents (‘vertex-dominated grain growth’) (Fullman 1952, Kawasaki, Nagai and Nakashima 1989, Nakashima, Nagai and Kawasaki 1989). Several newer physical systems have also been studied, such as magnetic bubbles (Molho, Gouzerh, Levy and Partesel 1986, Babcock and Westervelt 1989, Babcock, Seshadri and Westervelt 1990,
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A real experimental planar froth is made by sandwiching soap foam between two parallel plates (a Hele–Shaw cell) and draining out all excess fluid. It can be modelled without compromising the physics much, by observing that there are two very different time scales in the dynamics.

First, there is a fast mechanical time scale (surface tension driven), which minimizes the total length of the network while keeping constant the areas of the domains. It achieves a static equilibrium on which

1. the walls are a planar embedding of a planar graph,
2. this planar graph has a connectivity of three (higher connectivity is unstable),
3. walls are minimal surfaces (circular arcs),
4. the curvature of a wall is proportional to the difference in pressure across it, and
5. walls meet at 120° from each other at vertices.

The connectivity of the walls of the froth is three (except during topological changes) because any vertex with higher connectivity does not minimize total network length (Plateau 1873), as seen in fig. 1. We shall return to this point later.

There is a much slower time scale of a diffusive nature. Residual pressure differences between adjacent cells cause gas to flow through the membranes, from high-pressure cells to low-pressure cells. Noting that the height of the cell is a constant and that the gas is incompressible at the scale on which the tiny films can exert pressure, the amount of gas inside a cell is proportional to its area, and the amount of area flowing through a given wall is proportional to its length and the pressure difference across it. The product of the length of a wall and the pressure difference is proportional to the angle which the wall subtends with respect to its centre. Thus the total amount of area flowing in or out of a domain is proportional to a sum of subtended angles. With a little ingenuity, and using the above, one can show that the total rate of change of the area of a domain is

\[ \dot{A} = k(n - 6), \]

which is known as the von Neumann (1952) law. Here \( k \) is a rate constant depending upon surface tension and diffusivity, and \( n \) is the number of sides of the domain. This law is exact under the assumptions. There is no dependence on the areas, or on the local configuration other than through \( n \), the coarsest topological quantity in the system. The quantity \( n - 6 \) is called the topological charge of the bubble. The reason for this is that its sum over the network is a constant, independent of the network itself, by virtue of the Euler theorem. Topological charge cannot be destroyed; it can only be redistributed.

Fig. 1

A fourfold connected vertex decays into two threefold vertices because network length is reduced.
Fig. 2

Five sided bubbles have walls which curve outwards, while seven-sided bubbles have walls that curve inwards.

Fig. 3

Charge scattering processes. (a) Disappearance of bubbles (the 3 and 5 sided cases are shown, the 4 sided case is analogous). (b) Side swapping.

From eqn. (1), we see that five-sided bubbles shrink and seven-sided bubbles grow, while six-sided bubbles may change their shape but not their area. We can intuitively understand this by remembering that internal angles are supposed to be 120°, and so the walls of five-sided bubbles must be on average curved outwards, while the walls of seven-sided bubbles must be curved inwards (fig. 2).

Three-, four- and five-sided bubbles vanish in finite time. The total number of bubbles in the experimental cell decreases, and hence the average area increases. When a five sided bubble shrinks to disappearance, a fivefold vertex is produced and then decays; local topology is obliged to change. The negative charge in the fivefold bubble is redistributed, as shown in fig. 3. Other bubbles become negatively charged and in turn vanish. This proceeds ad infinitum, the froth coarsening and coarsening. As mentioned before, Smith conjectured that the froth would eventually reach a scaling state, a state on which successive configurations would be statistically similar except for average length scale. Dimensional analysis indicates that, since the only dimensional quantity in the formulation is the von Neumann k, the scaling should be $\langle A \rangle \approx r^1$.

1.3. The phase space

Given a planar graph A, we can define its dual B through the following procedure (Preparata and Shamos 1985, Fortune 1992). To every facet of A we assign a vertex of B; we join vertices in B whenever the facets in A share an edge, thereby assigning edges to edges. As a consequence, vertices of A are assigned to facets of B, and, if the procedure is performed twice, we obtain the original graph. There is one duality relation which will be relevant for our purposes here: the dual graph to a connectivity-three graph is a triangulation.
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Fig. 4

Several embeddings might be possible for some graphs. In both (a) and (b) the graph is the same; but in (a) cell A is adjacent to cell B while in (b) A is adjacent to C. This graph fails to have a well-defined dual, since cells A and C share two edges rather than only one.

I shall choose to describe the topology of the froth as a triangulation, rather than as a connectivity three graph, because of mathematical convenience. While prima facie equivalent formulations and trivalent graphs are obviously connected to the physics, triangulations are more convenient for the structure of the theory. The problem is associated to very technical caveats which I shall outline now only in broad terms. First, duality is defined only through an embedding; the facets of the graph intervene in the definition. Thus, if several embeddings were possible, there would be more than one dual graph (see fig. 4). A dual graph may not exist; whenever two facets share more than one common edge, the dual ceases to be a graph in the sense above, there being more than just one connection between vertices. Such structures may exist, but it has been shown that soap froth does not spontaneously produce them and they are killed in finite time by the dynamics (Weaire and Kermode 1984). It is therefore convenient to work in a framework that does not allow them from the start. The second reason is that we are going to be concerned with the areas of bubbles. In a triangulation, the bubbles are vertices, and areas are assigned to vertices. In a trivalent graph, the bubbles are facets (which a priori require an embedding), and the areas are numbers defined on these facets. Unfortunately, defining an area correctly in terms of a facet requires an orientation of the edges around the facet, an additional structure which we shall not need at all in the triangulation framework. It is therefore possible to propose the theory in the triangulation framework without ever having to declare the triangulation to be embedded in any fashion. I should state that patterns which do not have a dual do appear in nature; in particular, two-sided bubbles have been observed in domain growth in metals. It is entirely possible to extend the framework that I present to that case, but the extra degree of arid care that it requires does not warrant doing so for the purposes of this paper.

It can be shown that the states of the froth can be locally parametrized by topology and areas alone (Magnasco 1992); whether this is globally true is still under investigation (V. Fradkov and D. Udler 1992, private communication). In any event the phase space of froth configurations consists of a triangulation and a scalar field (the areas) defined on the vertices of this triangulation. We shall call the procedure that reconstructs the full shape of the froth from this information the froth embedding.

The scalar field evolves according to the von Neumann law (eqn. (1)); the triangulation changes when the set of areas is such that an appropriate embedding is no longer possible, because either an area or the length of a wall would become negative. For any given topology, there will be an open set in the space of all areas where an appropriate embedding can be found; when the von Neumann law moves the state of the system in this open set and collides with the boundary, then topology changes and there is a new different set of allowed areas.
If we alternatively wish to concentrate on topology, we may see that the disappearance processes look much like a spin-blocking operation. In the Ising model, the real-space renormalization group (RG) is implemented by performing all spin blockings in parallel. Bubble disappearance processes (and side swaps) do not commute, and so they cannot be carried out in parallel. It is necessary to provide a time-ordering operator, a *schedule* that says which blocking occurs first. This schedule is handsomely given to us by von Neumann's law. We can then interpret the dynamics of soap froth as a sequentially implemented spin-blocking procedure, as a kind of RG dynamics. RG has been mostly used as a tool for getting rid of length scales that bother the theorist in his formalism; in the case of froths, it is not a tool but the real dynamics of the system, which it uses to get rid of very real length scales; it is the mechanism of coarsening. From our experience with the RG formalism, it would not seem surprising if the system evolves towards a stable fixed point of renormalization, in froth language a scaling state.

§2. SELF-CONSISTENCY

Triangulations are beautiful objects, and they possess an idiosyncrasy which is to be respected. They are quite unlike spin lattices, for the value of topological charge is not a free variable which can be set independently of neighbour values; it *defines* who the neighbours are. Suppose that we wanted to increase by one the connectivity of a vertex. This means there should be one more link coming out of that vertex. However, the link has to go somewhere; so we need to increase the connectivity of another vertex. Thus we have added one more link to the triangulation but, since a triangulation is a saturated planar graph, meaning that no more links can be added without losing planarity, we have just lost planarity. Hence we need to remove some other link in order to recover planarity. This link goes between two vertices; hence we have to decrease their connectivity. Therefore changing one topological charge involves changing at least four charges.

Thus any local change in topological charge affects the connections to the rest of the triangulation; nearby changes do not commute. There are many restrictions related to what I shall call self-consistency: one always has to ensure that the triangulation is really a planar graph. I shall argue that this is really an important issue through a simple *counting* argument.

The number of different triangulations with $v$ vertices is (Agishtein and Migdal 1991)

$$T(v) = \frac{6}{v-4} \binom{4v-11}{v-5} \approx \frac{1}{2081.6} (4^v 3^{-3}) v^{-3/2} \exp\left(\frac{8}{3v}\right).$$

If we count the number of different configurations of a system of $v$ independent spins having integer values in $[-3, \infty)$ and subject to just one constraint (the sum over all spins is zero) we obtain

$$U(v) = \binom{4v-1}{v-1} \approx \frac{1}{8.6859} (4^v 3^{-3}) v^{-3/2} \exp(-\frac{3}{2}v^{-3/2}).$$

The quotient between these two numbers does not approach a constant as $v \to \infty$:

$$\frac{U(v)}{T(v)} \approx \frac{219}{3} (v-4) = 239.729(v-4).$$
Hence, topological charges are subject to additional constraints besides overall neutrality. A naive ‘independent connectivity’ argument would lead us to think of a phase space which is much larger than it actually is. These constraints mean that there are correlations between topological charges, which just cannot be avoided.

I shall call these self-consistency correlations. In studying them, a geometric construction will be useful.

2.1. The geometric analogy

To every triangulation we can associate a polyhedral surface, by requesting that all links be unit length and, naturally, all triangles be equilateral. Such a surface cannot be flat; if we attempt to glue five equilateral triangles at a single vertex, we shall notice that we produce a pyramid, for the amount of boundary provided by the triangles is too small for the ‘radius’ of the construction. Conversely, if we glue seven triangles we get a saddle-like shape, for there is too much boundary. Six triangles is the divide and produces something flat (fig. 5). This is a natural mapping from the triangulation to a Riemannian surface; we can do complex function theory on this surface (Bers 1958, Voevodskii and Shabat 1989). This surface is naturally a (singular) Riemannian manifold too (Bers 1958, Sadoc and Rivier 1987); curvature is concentrated at the vertices of the graph, which have become conic singularities.

We shall now show that curvature is proportional to topological charge. If a vector is parallel transported along a loop, the angle through which it rotates is equal to the integral of the curvature inside the loop (Wald 1984, Boothby 1986). If we perform this procedure on a cone, this angle equals zero for any loop not containing the tip of the cone and equals the defect angle of the cone for any loop containing the tip; curvature is thus concentrated on the tip, in ‘Dirac delta’ fashion, and the amplitude of this Dirac delta is the defect angle. If we glue equilateral triangles together at a tip, the defect angle equals 60° (π/3) times the number of triangles minus six. The sign of the curvature can be ascertained from fig. 5. Thus the amount of curvature concentrated at a vertex equals 

\[ \int_S R \, da = -\frac{1}{3}\pi Q, \]

where \( S \) is a piece of surface containing only the vertex in question. Integrating over the whole surface \( M \), we obtain the equivalence between the Euler and Gauss-Bonnet theorems:

\[ 4\pi(1-g) = \int_M R \, da = -\frac{1}{3}\pi \sum_v Q_v = \frac{1}{3}\pi \times 12(1-g). \]

Triangulations do not approximate arbitrary Riemannian manifolds in a smooth sense but can approximate them in an integral sense (by integrating with respect to test

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**Fig. 5**

Glueing equilateral triangles together. (a) Five triangles give a pyramid: positive curvature. (b) Six triangles is flat (zero curvature). (c) Seven triangles gives a saddle (negative curvature).
functions which contain lots of vertices). They are also dense in an algebraic sense; this is explained in minimal detail in Appendix A.

This mapping is a particular instance of a more general way to discretize Riemannian manifolds, which is called Regge calculus (Misner, Thorne and Wheeler 1973). It has been the basic numerical tool for classical general relativity for many years. In higher dimensions, triangles are replaced by simplices (tetrahedra in three dimensions), and triangulations are replaced by simplicial complexes, topological entities obtained by gluing simplices along identical triangular faces. In this more general scheme, the sides of triangles are allowed to assume any length which is compatible with the triangle inequalities. Curvature remains a defect angle, as above, but it is now concentrated on the $n-1$ dimensional elements of the complex; it is no longer quantized, even in the two-dimensional case, because the lengths of links are not quantized. They are not smooth objects, and calculations which use them work only in an integral sense ('spectrally accurate' code), usually by writing the equations of motion in integral form, from a minimum principle.

2.2. Self-consistency correlations

From the geometrical analogy we immediately deduce that the number of neighbours of a vertex suffers the same kind of correlations and constraints as a curvature field.

A Riemannian manifold is defined through a metric tensor. This only needs to be positive definite, and to satisfy the global topological constraints that tensors have to satisfy on a differentiable manifold. From this metric tensor, one constructs the Kristoffel symbols, and then one can take covariant derivatives. A suitably constructed second derivative of the metric tensor is the full Riemann curvature tensor (four indices, and nonlinear in the metric), taking two traces, one obtains the Riemannian curvature field. A typical scalar field is not a curvature field, for the nonlinear differential equations tying the metric to the curvature will not have a positive-definite metric tensor solution for an arbitrary curvature field (Gromov 1986).

So we expect the topological charges in a triangulation to satisfy many constraints too. The most obvious can be stated in a purely graph-theoretical manner: each subgraph is obliged to satisfy the Euler theorem. The content becomes more apparent when we look at it in terms of curvature.

A convex piece of a curved surface cannot accumulate too much positive curvature. If the curvature inside such a piece becomes $4\pi$, then a sphere will have been closed and detached from the rest of the surface (fig. 6). Similarly, no convex subset of a triangulation can have a total topological charge in excess of $-11$. This is because, as soon as the total charge becomes $-12$ (the integral of curvature equals $4\pi$), a sphere has

Fig. 6

As we accumulate too much positive curvature together, a piece of the surface 'buds'. (a) $2\pi$ of curvature together make an igloo; (b) $4\pi - \epsilon$ of curvature make a small sphere attached through a narrow neck.
been closed and therefore has detached from the triangulation, which is no longer connected, in contradiction with the hypothesis.

There are many inequality constraints on the charges of the triangulation, one for each convex set (and there are many). The consequences of this fact haunt mean-field theories.

2.3. Mean-field theories

Several mean-field theories have been built for soap froth and related systems. The usual methodology is the following: we consider the joint probability distribution \( \rho(A,n) \) of areas and number of sides.

We then attempt to write a master equation for \( \rho \). The von Neumann law acting on the ensemble can be stated trivially: \( \dot{\rho} = k(n-6) \partial_n \rho \). The boundary conditions are not trivial; that is, what should be done with the probability flowing through \( A = 0 \) from the right, which corresponds to the vanishing processes. Charge redistribution couples then the different \( n \) in some fashion which is far from self-evident. Side-swapping processes introduce further complications and are usually neglected. In order to get anywhere with this approach, one is forced to assume that there are no neighbour correlations.

The conceptual flaws of the construction are clear then. Writing down \( \rho \) is akin to printing on paper an image of a froth and then cutting along all edges to disassemble the pattern into a set of polygons which are then histogrammed. The inverse problem is much harder; putting a froth together from some collection of polygons is a puzzle which in general will have no solution.

However, the assumptions of mean field theories do specify an algorithm for solving the puzzle; if there exist no neighbour correlations, then the froth is to be reassembled in uncorrelated fashion. We should be able to do this if the whole ‘theory’ is to be self-consistent at all. I shall show now that this cannot be done, thus proving that such mean-field theories are not self-consistent.

We shall deal only with the topology, for, if we cannot deal with the topology alone, we certainly cannot deal with both topology and areas. We shall attempt to reconstruct the triangulation piece by piece, by layers. We shall first put down one vertex, then its first neighbours, then the second layer of neighbours and so forth. As we do so, we notice that eventually the triangulation closes on itself like a sphere (V. Fradkov and D. Udler, private communication, 1992), the boundary becoming smaller and smaller, until there is only room for just one vertex. If the new vertex happens to have the right number of neighbours, we close the sphere; otherwise we cannot glue it. In either case the attempt has failed at a finite step in the construction.

We can perform this procedure numerically and histogram the layer number at which the construction failed, to obtain fig. 7. It is clear from the figure that the tail is an integrable power law, so that the probability of failure is actually one. We can further understand this analytically, by noting that the number \( C_n \) of vertices at layer number \( n \) depends on the total topological charge \( Q_n \) at the said layer through the equation

\[
C_{n+1} - 2C_n + C_{n-1} = Q_n
\]

(we shall return to this equation in the next section).

For sufficiently large \( C \), the central limit theorem assures us that the \( Q \) will approach Gaussian random numbers, with a standard deviation of \( C^{1/2} \mu_2 \), the standard deviation of the single vertex distribution.
Since we are interested in the probability of success, which happens when $C_n$ becomes large as $n \to \infty$, we can take the continuum limit of this equation to obtain a Langevin equation:

$$\frac{d^2 C_\rho}{d\rho^2} = \xi C_\rho^{1/2},$$

with $\xi$ a Gaussian random function of variance $\mu_2$. Superficially the scaling of the equation might seem to imply that $C \approx \rho^4$, which would indeed be appropriate for a mean-field theory, but the equation actually has finite time singularities with probability one, for, changing variables to $C = y^2$, $t = \rho$, we obtain

$$\dot{y} = -\frac{y^2}{y} + \frac{1}{2} \xi,$$

which is an equation for a random walker in a logarithmic potential. The probability that the random walker reaches a distance $y$ is algebraically decreasing and the probability that it escapes to $\infty$ (the 'success' of the construction) is precisely zero. $C$ will eventually become zero again, a sphere will have been closed and pinched off, and the construction will have been terminated in a finite number of steps, in full agreement with the numerical evidence.

Thus, if we assume that there are no correlations at all, we might obtain some distribution as a solution to the master equation, but, in order to construct an actual example of the topologies in question, we need to assume correlations, violating gravely the self-consistency of the process.

The same happens if we assume that correlations are cut off (or become negligible) beyond one domain, for then we just have a coarsened version of the above argument.
No mean-field theory of this form can take into account the self-consistency correlations which, although perhaps weak, are certainly infinite ranged. Even then, these correlations do not prescribe what the long-range properties of the froth are; they just ensure than the triangulation is a planar graph.

The origin of this problem lies with the methodology of mean-field theories. It is certainly the case that the number of sides of the domains is the prime observable, and that global properties such as intrinsic dimensions (which we shall study in the next section) are hard to compute or measure experimentally. Even then this does not mean that the mean-field theory should be built around the probabilities of topological charges. In the case of geometric theories, such as classical general relativity (Wald 1984) or interface dynamics, fluctuations are taken into account by linearization of the equations of motion, and they are written for a perturbation $\delta g$ of the metric tensor $g_0$. The equations are written for the total $g = g_0 + \delta g$, all higher-order terms in $\delta g$ are discarded, and then the (now linear) equations are solved; only after this process is the curvature field computed.

The equations could not be written for the curvature field directly since they would be highly nonlinear integrodifferential equations with infinite-ranged interactions, and it would not be at all clear what the correct process to linearize would be. The concept of the background metric would be lost and therefore the global properties of the solution could not be ascertained. Yet, this is basically the type of methodology usually employed for froth mean-field theories: use of the curvature as the dynamical variable rather than a metric tensor or, even worse, use of only the probability distribution of curvatures.

We could envisage thus, a 'correct' methodology for writing mean-field theories. We could try to separate the topology between a background piece and a fluctuation piece giving the local variations. Loosely coupled equations could then be written for both pieces. In §4 we shall derive the equation for the background, showing that von Neumann's law generates a flat background metric. How to derive an equation for the fluctuation piece is still unclear to the present author.

§ 3. INTRINSIC DIMENSION

3.1. Definition and relevance

Given a point $v$ on a triangulation, I shall define the disc of radius $\rho$ centred at the vertex $v$ as the set of points which can be reached from $v$ by jumping at most $\rho$ links, and the area of this disc as the total number of points contained inside. Then, for a regular triangular lattice on the plane,

$$A(\rho) = 3\rho^2 + 3\rho + 1$$

independently of $v$. It is obvious that it increases quadratically with $\rho$.

I shall then define the intrinsic dimension of a triangulation to be

$$D = \lim_{\rho \to \infty} \left( \frac{\log \left[ A(\rho) \right]}{\log \rho} \right).$$

(In this limit, $D$ becomes independent of the centre $v$.) Why should one be interested in this quantity? All properties of a statistical mechanics system living on the triangulation will be affected by it, for the RG formalism depends most essentially on the way that the density of eigenstates diverges near the zero wave-vector. This is a function of $D$. Therefore the overall behaviour of a spin system or a field theory on a triangulation is controlled, above all, by this number.
3.2. Elementary examples

Obviously, \( \mathbb{R}^n \) and the regular lattices in it, such as \( \mathbb{Z}^n \) (the lattice of vectors with integer coefficients), have intrinsic dimension \( n \). However, as soon as the system loses scale invariance, \( D \) will differ from the topological dimension of the space. For instance, \( D = 0 \) for a sphere or any compact surface, and \( D = 1 \) for a cylinder, even though it is locally a two-dimensional set. The way that this affects statistical mechanics is straightforward; a spin system having local interactions cannot have a phase transition at all on a compact lattice (\( D = 0 \)), for the partition function is a finite sum of exponentials and hence is analytic. It would not have any \( T \neq 0 \) phase transition on a cylinder (\( D = 1 \)), for, as soon as the domains become as large as the circumference of the cylinder, the system stops being self-averaging and the system behaves effectively as a one-dimensional lattice. This phenomenon (that we may add further dimensions and still obtain a lower-dimensional effective behaviour) is called compactification.

3.3. Hyperbolic space and hierarchical structures

Hyperbolic space \( H \) gives us a more interesting example. Defined as the unit disc in the complex plane endowed with the line element

\[
dS = \frac{|dz|}{1 - |z|^2},
\]

it is a non-compact, simply connected Riemannian manifold with constant curvature equal to \( -1 \) (Series 1987). The geodesics on this space are arcs of circle which intersect the boundary at \( 90^\circ \); the isometries (the equivalent of the Euclidian roto-translations) are Möebius maps:

\[
\tilde{\eta}(z) = \frac{az + b}{bz + \bar{a}}, \quad |a|^2 - |b|^2 = 1.
\]

The space is infinite, and any point on the circle is infinitely far away from any other point; the geodesic distance from any point to the centre is

\[
\rho(z) = \tanh^{-1}(|z|).
\]

The area of a circle of geodesic radius \( \rho \) equals

\[
A(\rho) = 2\pi(\cosh \rho - 1),
\]

which agrees with \( A \approx \pi \rho^2 \) for \( \rho \ll 1 \) but increases exponentially as \( \rho \to \infty \). Therefore \( D = \infty \). Any statistical mechanics system living on a regular lattice embedded in \( H \) (there is an infinite number of them) will behave near phase transitions exactly like its mean-field theory formulation.

From \( H \) it is possible to obtain all compact manifolds of genus greater than unity by elementary surgery (Poincaré 1882, 1895, Klein 1890) (fig. 8). For instance, glueing alternate sides of a hyperbolic octagon with \( 45^\circ \) internal angles (provided of course that we have given the same length to the sides to be glued) we obtain a two-torus; equivalently, we may use the glueing isometry to map the octagon to obtain a tesselation of hyperbolic space by octagons, eight such octagons meeting at one corner. This particular tesselation has been extensively illustrated by Escher in his Circle Limite series.

It is also possible to obtain a vast amount of non-compact manifolds (Hadamard 1898); simple recursive surgery leads very easily to hierarchical structures. For instance, a tree structure (the manifold given by the bark of a real tree if this were smooth) can be
Surgery: gluing the sides of a hyperbolic octagon with 45° internal angles to make a two-torus. An $n$-torus may be obtained by similarly gluing a $4n$-gon with $\pi/2n$ internal angles.

![Fig. 8](image)

Surgery of $H$ to make a ‘tree’: (a) identifying sides of an octagon; (b) gluing gives a ‘pantalon’, a surface with three holes; (c) recursively gluing pantalons we can make tree structures.

obtained by cutting and gluing to create a surface with three holes, and then gluing these to form a tree, as shown in fig. 9. This surface still has $D = \infty$.

Regular triangular lattices can be constructed in $H$ for any connectivity higher than six. Thus we get a complete classification of all simply connected ‘homogeneous’ triangulations. There are three finite triangulations with connectivities three, four and five: the tetrahedron, the octahedron and the icosahedron. These three plus their three duals are the five Platonic solids (the tetrahedron is self-dual). There is one homogeneous, simply connected triangulation with connectivity six: the regular triangular lattice. Then there are an infinite number of triangulations of arbitrarily large connectivity, such as triangulations of hyperbolic space. All these lattices have the additional structure of being Cayley graphs of isometry groups.

Subgraphs of these lattices can be trees. For instance, a regular rootless binary tree can be embedded inside a connectivity-nine lattice as a fully symmetric lattice; these tree graphs (Cayley trees or Bethe lattices) have been for a long time the paradigm of infinite-dimensional lattices.

In all these, the exponential growth of areas implies the exponential growth of the boundary, and therefore there is always as much surface as there is bulk. If subexponential growth were to occur whilst keeping $D = \infty$, then one could have more bulk than surface and a true thermodynamic limit.

3.4. Random triangulations and random trees

There are enumeration results for triangulations (Tutte 1962, Brezin, Itzykson, Parisi and Zuber 1978). This enumeration is usually performed using Schwinger–Dyson equations. The number of planar triangulations with $N$ triangles bounded by $L$ edges is

$$T(n, l) = \frac{3(l+2)(l-1)!}{3(l+n+1)!} \sum_{j=0}^{m} \frac{(4n+3l+1-j)!(l+j+2)(l-3j)}{j!(j+1)!(l-j)!(l-j+2)(n-j-1)!},$$

where $l = L - 3$, $n = \frac{1}{2}(N - L) + 1$ and $m = \min(n-1, l)$. 

Correlations in cellular patterns
Fig. 10

A typical triangulation of the torus. 2500 vertices; starting from a regular triangular lattice, 15000 edges were scrapped.

An interesting question to pose is what does the 'typical' triangulation look like. This question is well posed in the \( v \rightarrow \infty \) limit.

The typical triangulation looks like fig. 10. There are triangles and even whole patches nested inside triangles in a seemingly random way. In the geometric view, we would say that the surface sprouts fingers which recursively sprout fingers \textit{ad nauseam}, in a disordered fashion.

It is therefore a random tree, also called a 'branched polymer'. This surface is vital to string theory and two-dimensional quantum gravity. Both of these are geometric theories dealing with two-dimensional Riemannian manifolds, and, in both of these, the geometric identification which maps triangulations to Riemannian manifolds is a natural way to discretize the surfaces. In string theory (Billoire and David 1986, Agishtein and Migdal 1991), the basic propagator is written as an integral over all Riemannian metrics; when discretized, it becomes a sum over all triangulations. In much the same way that random walks dominate the path integral for a single particle, being the 'typical' path, branched polymers dominate the string theory path integral. In the case of two-dimensional quantum gravity (Agishtein and Migdal 1991), space–time is supposed to look like a branched polymer on the Planck length scale, for on that length scale it is fluctuating as much as it can.

Whenever we speak about correlations, we shall take consistency correlations for granted and keep in mind that the \textit{most uncorrelated topology looks like fig. 10}. Whatever correlations these surfaces have, they are just the self-consistency correlations.

We expect, from the previous discussion on tree structures, that random triangulations will have an infinite intrinsic dimension. This has been shown to be the
case by Agishtein and Migdal (1991); they (numerically) find that
\[ \ln [A(\rho)] \approx a + b \log \rho + c \log^2 \rho, \quad c > 0. \]

The reason for this divergence is simple. Imagine that we construct the surface of a tree by joining together branching sites (like the pantalon above), and capping off both the root and all the tips of the tree with hemispherical caps. If we choose a centre somewhere in the middle of the tree, a circle growing out of this centre will see only negatively curved pieces of surfaces for a long while, until it hits either the tips or the root. Therefore the circumference of this circle will be growing exponentially all the time.

If we examine what happens to a growing circle (as above) in a smooth, nearly flat surface, we find that it may deform, but it never stops being simply connected. However, if we imagine a tall cylinder attached in some smooth manner to a plane, we can see that circles originating far from the cylinder will, when reaching it, become 'entangled'; for sufficiently large radii, the circle will consist of two distinct disconnected components, one looking like a circle in the absence of the cylinder, and the other climbing up the cylinder. If instead of a cylinder we provide a branched structure, we shall observe further splittings into more disconnected components; if the branching is regular, then there will be an exponential explosion in the number of components, until some circles reach the tips of the branches. If (following the quantum gravity interpretation) we want to define the radius of these circles as time, and the circles themselves as space-like surfaces, then the disgregation into separate components is interpreted as the creation of baby universes. The creation rate of baby universes in random triangulations seems well defined (Agishtein and Migdal 1991).

Algorithmically, random surfaces are easy to make. A triangulation is kept in the computer's memory using the appropriate data structures (the doubly connected edge list (DCEL) in Preparata and Shamos (1985)). A link in the triangulation is picked at random, with equal probability per link. The link is then swapped. (In fact, one should check that swapping the link does not create a multigraph.) This procedure is repeated many times, until the statistics of the ensemble equilibrate. This procedure cannot be vectorized, but it can be parallelized through a very ingenious trick (Agishtein and Migdal 1991). Even on a single processor workstation, random triangulations with a few hundred thousands of links can be evolved within a few hours. Embedding the triangulation on the plane so that it can be drawn (as fig. 10 was) is a much more difficult proposition and it can be comfortably done only up to a few thousand vertices.

3.5. Random Voronoi construction

Given a set of points on the plane the Voronoi construction assigns to each a subset of the plane consisting of all points nearer to the point in question than to any other (Preparata and Shamos 1985, Fortune 1992). The boundaries of the domains of each point trace a connectivity-three graph (except for some symmetric cases). The dual of this graph is the Delaunay triangulation of a set of points; it has the property of maximizing the minimal angle in the triangulation.

Algorithmically, there is a 'fast' way to construct these graphs, which takes \( O(n \log n) \) time (Preparata and Shamos 1985, Fortune 1992).

We can randomly place points on the plane and look at the topologies generated. The simplest case is when we randomly place points on a torus with constant mean density, as shown in fig. 11.
I shall show (in §4) that uneven density distributions result in non-zero local mean topological charges (and therefore non-zero mean Gaussian curvature) associated with the changes in the density. The Laplacian of the logarithm of the density of points controls the mean curvature, positive Laplacians resulting in negative curvature and negative Laplacians resulting in positive curvature.

We would expect to have zero mean curvature on all regions in fig. 11, for the density is uniform and, therefore, an intrinsic dimension $D = 2$, corresponding to a flat surface. However, the mean density of points in this two-dimensional Poisson process has fluctuations, and these fluctuations are unbounded. In fig. 12, the average area of topological discs of radius $\rho$ is plotted, as a function of $\rho$, for a reasonably large ($10^5$ vertices) random Delaunay triangulation of the torus. All circles of radius $\rho \leq 60$ were constructed for all $10^5$ points in the triangulation. We can see that the area is always substantially larger than the Euclidean formula $A(\rho) = \pi \rho^2$, or the triangular lattice’s $A(\rho) = 3(\rho^2 + \rho)$; it eventually settles to what seems to be a power law just above quadratic; a least-squares fit to the log-log plot yields $D = 2.08 \pm 0.01$. I cannot ascertain whether this is genuinely a dimension larger than two or whether a logarithmic divergence $(A \propto \rho^2 \log \rho)$ is spoiling the fit, but the latter seems more probable. In any case, for circles with a radius larger than five, the area is more than double the Euclidean $\pi \rho^2$ or the triangular lattice’s $3\rho^2$ and it is still increasing at the end of the range of the data.

We can apply eqn. (5) to the data, to obtain the average topological charge on a circle of radius $\rho$. For the circle of unity radius, this can be expected to be negative, since given a $(+1, -1)$ pair of vertices (a disclination) the number of circles which see only the negative charge is larger than the number of circles that see only the positive charge, simply because the number of neighbours of the negative charge is seven whilst the
number of neighbours of the positive charge is just five. More precisely, the average charge on circles of unity radius is

$$\langle Q_1 \rangle = \frac{1}{N_{\text{total}}} \sum_{n=3} (n-6)nN_n = \sum (n-6)^2 P_n = \mu_2,$$

where $\mu_2$ is the variance of the number-of-neighbours distribution $P_n$. Since $C_2 = 12 + 2Q_0 + Q_1$, the average circumference of a circle of radius two is less than $C_2 > 12 + \mu_2$. A straightforward application of this is the Aboav–Weaire law. Assuming that the average circumference of a circle of radius two with an $n$-sided vertex at the centre is of the form

$$\langle C_2 \rangle = \langle C_2 \rangle + a(n-6),$$

we find that the average charge $F_n$ of the neighbours of $n$-sided vertices is

$$F_n = \frac{\mu_2 + 6(2-a)}{n} (2-a).$$

Applying eqn. (5) to the data reveals that the average charge on circles of all radii seems to be always negative, which explains the large divergence from Euclidean statistics. Also note that there is no contradiction between these charges always being negative and global neutrality, for circles are not generic sets.

It is overall difficult then to establish whether the random Voronoi construction is asymptotically flat or not. Since the probability of large fluctuations in the density of points is exponentially small in the number of points within the fluctuation rather than algebraic, we would expect a logarithmic divergence; but this is yet to be established.
A study of topological charge correlations between first neighbours was undertaken by Fradkov et al. (1987). They compared the slopes of the Aboav–Weaire relation in three different cases: the mean-field theoretical 'polygon gas' with no correlations at all, an actual experimental cellular pattern, and a 'most probable' topology with the same probability distribution $P_n$ of number of sides as the experimental pattern. They defined and constructed this 'most probable' triangulation with a given $P_n$ by taking a triangulation, selecting a link randomly, and swapping it if the effect on the distribution function was within some tolerance bounds.

This is a particularly clear example of how correlations in cellular patterns appear in several stages. What Fradkov et al. found is that the 'most probable' triangulation indeed obeyed the Aboav–Weaire relations, but the slope was about half that of the real cellular pattern, indicating that there was substantially less topological charge shielding in it.

What happens is that the 'most probable' triangulation in the sense used by Fradkov et al. is precisely the 'random triangulation' algorithm of discrete string theory that we discussed above, subject to constraints. We expect that for any generic $P_n$ it will produce random surfaces, only with a different cut-off scale. We immediately deduce that their intrinsic dimension will be infinity, since there is a change only in the scale of the tips.

Thus the correlations observed by Fradkov et al. were just the self-consistency correlations. The extra amount of correlations in the actual cellular pattern results from the globally flat structure of its topology; we shall consider this issue in more detail in §4.

### 3.7. The Fradkov–Shvindlerman–Udler–Beenaker model

This is a model with no embedding (or, perhaps, a trivial embedding); the areas are only constrained to be positive (no lengths of sides are tracked); evolution takes place by the von Neumann law and the charge redistribution processes are effected randomly (Fradkov, Shvindlerman and Udler 1985, Beenaker 1988, Fradkov and Udler 1990). Swap operations depend entirely on side information not present in the model; so Fradkov et al. introduced a rate to generate them; Beenaker's model is a 'zero-rate' version of that of Fradkov et al.

Fradkov et al. found an absolutely remarkable result: for any value of this rate (up to and including infinity) the system reaches statistical equilibration and a scaling state. This scaling state varies continuously as a function of the rate. All scaling states produce fewer neighbour correlations than typical cellular patterns, for all values of the rate, just as in the previous section.

This topological model converges to string theory in the limit $rate \to +\infty$. Therefore for sufficiently large rates we shall expect branching of the surface. It is not trivial, however, to see that this is still true for small and zero rate. What happens is that this model does not require absolutely any a-priori correlation between areas and topology, because there is no embedding. Therefore there is no flat target space metric to refer to. The disappearance processes occur with a random orientation, independent of the local value of the areas; so the time evolution actually does not enforce any further correlations; whatever correlation is seen is just the effect of the von Neumann law acting on the ensemble and therefore shifting the average area as a function of topological class.
§4. EXTRINSIC AND INTRINSIC PROPERTIES

A surface in three-dimensional space inherits, from the Euclidean structure of the space, a Riemannian structure. One may define a metric tensor and a curvature for the surface (Boothby 1986). These are called extrinsic, for they are a function of the way that the surface is embedded in this larger space rather than being functions of the surface only. When we are given an abstract definition of a surface and a metric tensor on it, we say that the latter is intrinsic for it does not depend upon any ambient space. Mean curvature is a typical extrinsic measure, since it requires computation of distances in the embedding, while Gaussian curvature is intrinsic; it can be measured through deviations of angles within the surface.

The interplay between intrinsic and extrinsic properties is a complex subject and very much the target of current research. To give just a flavour of what is involved, imagine a tree with a very smooth bark; we shall take this bark to be our surface. Ants are blind and cannot jump, and hence while they are crawling around they certainly only perceive intrinsic properties of this surface, while monkeys or squirrels, who can see and jump through the ambient space, can see that branches which are intrinsically far apart (according to geodesic distance) can be quite close together in the ambient space. This is a major problem in interface motion, for, if one writes the interactions between elements of the surface, they propagate through the ambient space, and thus even shielded interactions which are mostly local in this ambient space become highly non-local in terms of geodesic distance. This interplay also gives its flavour to biological problems such as protein folding, for proteins self-contact and interact between points which are far apart in the genetic sequence.

There is another problem. Given a surface in three-dimensional space we obtain a metric tensor, but not every metric tensor can be realized by a surface in three-dimensional space (Gromov 1986). For instance, we may define a 'flat torus' as a flat rectangle with periodic boundary conditions, but this flat torus cannot be realized as a surface in three-dimensional space, only as a surface in four-dimensional space. Therefore embeddings in an ambient space may act as filters, which prune the space of all possible metrics to leave, perhaps, just a small subset of the space.

So far I have just mentioned isometric manifold embeddings. There are also other types of embedding related, for instance, to graphs. In these more general meanings of embedding, we have the following: an object \( O \), a 'target' space \( T \), and some mapping \( \phi : O \to T \). The object \( O \) is endowed with some structure \( \mathcal{O} \) and \( T \) with \( \mathcal{T} \); we shall say that \( \phi \) is an embedding of \( O \) in \( T \) if and only if the projection of \( \mathcal{T} \) on \( \phi(O) \) equals \( \mathcal{O} \) (through \( \phi^{-1} \)). Before, \( \mathcal{O} \) and \( \mathcal{T} \) were the set of open sets; we can now assume more general structures. Thus a graph embedding is a procedure which associates coordinates in a target space with each vertex and simple continuous curves with the edges, in such a way that curves corresponding to different edges do not intersect in the target space. Not every graph can be embedded in the plane, for instance; this is an instance of the 'pruning' mentioned in the previous paragraph. In fact, we define planar graphs as precisely those that can be embedded in the plane; likewise torus graphs are those that can be embedded in a torus.

The procedure which, given a triangulation and an allowed set of areas on the vertices, gives us the image of the froth is certainly, according to these notions, an embedding. There is a structure being preserved; we are performing a graph embedding, we are making the areas on the triangulation coincide with the bubble areas, and we are satisfying constraints (embedding through circular arcs and 120° angles). There is pruning; not every set of (positive!) areas is allowed, since some may
produce negative lengths of edges, and so the graph itself would not be correctly embedded.

A less restrictive notion than embedding is that of immersion (Guillemin and Pollack 1974, Boothby 1986). Immersions are only required to preserve structure locally; in the case of manifolds, for instance, the mapping has to be one to one locally, but self-intersections are allowed.

4.1. The embedding theorems

We shall review now some classical theorems related to embeddings. There are two types of manifold embedding: purely topological and isometric. There is also an embedding theorem for planar graphs. Unfortunately, we do not yet have a rigorously proven embedding theorem for froths, despite a substantial amount of work along these lines (Magnasco 1992, Fradkov and Udler 1993).

In topological embedding (Whitney), every differentiable manifold of dimension \( n \) may be embedded in \( \mathbb{R}^{2n} \) (Guillemin and Pollack 1974). By embedding, we refer to a procedure mapping the abstract blueprint of the manifold to an actual submanifold of the target space homeomorphically, that is self-intersections and ‘accumulation points’ are forbidden.

This embedding is purely topological and not geometrical. The surface of any doughnut is an embedding of a torus; both a soccer and rugger ball are embeddings of a sphere. We may request to embed a Riemannian manifold in such a way that its intrinsic (built-in) metric tensor is equal to the extrinsic metric tensor acquired through the embedding. This can be done according to a theorem by Nash.

In isometric embedding (Nash and Nash–Gromov–Gunther), an \( n \)-dimensional Riemannian manifold can be embedded isometrically in \( \mathbb{R}^{n^2+n} \). For orientable Riemannian surfaces, this can always be achieved in \( \mathbb{R}^5 \). (For target spaces of smaller dimensionality, only a small subset of the metric structures can be embedded isometrically (Gromov 1986).)

In graph embeddings (Halmos), planar graphs can always be embedded in the plane using only straight segments (Preparata and Shamos 1985).

4.2. Embedded random triangulations

An embedding for a triangulation on a compact Riemannian manifold is possible using a Poisson equation; we consider the links between vertices to be ideal Hooke springs along geodesics and requests an equilibrium position. Figure 10 was done in this fashion, on a flat square torus. On a sphere, the solution is always unique except for rotations. On a torus, we have to be more careful, for there are solutions to the Poisson equations which are not embeddings; different links may cross because there are many different geodesics which join two given points.

We note that the size of tips is much smaller than the size of root regions. This is because solutions to the Poisson equation achieve their extrema at the boundary, and hence the positions of the vertices in the finger are constrained to lie inside the narrow triangular neck through which it is attached to the rest of the triangulation; this induces exponentially decreasing sizes for the successive levels of branching of the surface.

4.3. Embedding-induced correlations

In a Delaunay triangulation or a froth embedding, there is a rather stringent correlation between the mean local area (i.e. an area averaged over a patch of bubbles) and the mean topological charge on that same region. We shall call \( a \) the local average
area, and \(\bar{a}\) the overall average area. If we think in terms of the underlying graph and its associated polyhedral surface, I shall argue that (under technical conditions on the amount of stress present, valid for Voronoi constructions and for freely evolving soap froth but not for stressed or rheological foams) it is described by the Riemannian metric

\[
g_{\mu\nu} = a^{-1}\delta_{\mu\nu}.
\]

Then the Christoffel symbols (Wald 1984, Boothby 1986) are

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(\delta^\alpha_\mu \partial_\nu \ln a + \delta^\alpha_\nu \partial_\mu \ln a - \delta^\alpha_\mu \partial_\nu \ln a) \ln a,
\]

the contracted Christoffel symbols are

\[
T^\alpha_\mu = \partial_\nu (\ln a)
\]

and the Ricci tensor is

\[
R_{\mu\nu} = \partial_\nu \partial_\mu (\ln a),
\]

from where the scalar curvature is \(R = \Delta (\ln a)\) and the mean topological charge is

\[
\bar{Q} = -\frac{3}{\pi} a \Delta (\ln a).
\]

So patches where the average area is smaller than the rest have a negative mean topological charge. This correspondence loosens from the exact relationship (8) to just a correlation saying that, if a bubble is smaller than its neighbours, it is probably negatively charged; this correlation is known as the Lewis law (Glazier 1989).

In the case of stressed or rheological foams, the stress is shown as a local anisotropy in the lattice; bubbles are more elongated along a certain direction than others. In this case the isotropic (conformal) metric given by eqn. (7) fails to represent reality, and we have to use a generic (quasi-conformal) metric tensor, whose eigenvector with larger eigenvalue corresponds to the direction of the stress. Soap froth will support only very small stresses elastically; side-swap processes will occur if larger amounts of stress are involved. Large stresses can be stationary only if the boundaries of the froth are constantly being sheared, as in flowing or rheological foams. I shall not treat this case here; the reader should refer to Appendix A for the general mathematical background on this issue.

There is a more local correlation which exists between area and topology, when we have the occurrence of nested structures. We saw in the last section that a Poissonian embedding maps nested structures into features with exponentially small areas. This also occurs in the case of froth. Herdtle and Aref (1991) have given a very elegant example of a (regular) fractal froth. They take a threefold vertex and replace it with a threefold bubble. This destroys the original vertex but creates three new vertices. The procedure is then repeated recursively for each of these new vertices, to obtain a fractal very much like the Sierpinski gasket. When we assign areas to the newly created bubbles, we have to make sure that they do not intersect each other, and that they leave enough room for the bubbles in the level next to them. Herdtle and Aref showed that this forces geometrically decreasing areas for the bubbles in each level.

**4.4. Dynamics-induced correlations**

We have just seen that there are two ways in which the froth embedding induces correlations between topology and areas. These correlations are 'static', in the sense that a suitable set of areas can be found for absolutely any topology. However, as
dynamics act upon the system, some topologies will fail to recur; dynamics act upon the statically correlated states to produce much more pronounced correlations.

There are three processes involved. First, the von Neumann law acting upon 'branched' triangulations will destroy the inclusions in finite time. Second, the charge-transfer processes that occur when a bubble disappears have a definite 'polarity', in the sense that charge will be transferred to the neighbours according to their area. Thus the froth does not possess any mechanism for the rebuilding of branched-polymer structures, and these will just never recur. So there is only global structure to worry about. We shall show in the following section that a flat background metric is stable under the time evolution (on assumption of the existence of a scaling state).

Herdtle and Aref (1991) showed that, for the regular Sierpinski gasket type of froth with perfect fractal scaling we described in the previous section, the von Neumann law destroys the 'decoration' in a finite time which is completely independent of the number of levels of nesting of the structure. Somehow they fail to emphasize a point which happens to be important: the time required for the destruction of the gasket is smaller than the time for the disappearance of a triangular bubble of the same area. We usually think of triangular bubbles as the most ephemeral structure on froth, because their \( \hat{A} \) is so negative and they are usually small, but fractal decorations of this type are actually killed faster.

It is trivial to show that this is the case for absolutely any decoration structure, fractal or not. The total topological charge of a piece of triangulation nested within a single triangle (a vertex decoration on the dual trivalent graph) equals minus the number of edges connecting it to the triangle; this number is necessarily larger than or equal to three, and therefore the total \( \hat{A} \) of the decoration is negative, and larger than or equal to in absolute value that of a triangular bubble. Strict equality is only attained for a triangular bubble. Thus the von Neumann law kills these structures rather fast, firstly because \( \hat{A} \) is large and negative, and secondly because their initial area was necessarily small to begin with.

It has been shown by Fradkov, Magnasco, Udler and Weaire (1993) that five-sided bubbles can disappear as five sided, that is that they need not lose sides while shrinking. When a five-sided bubble vanishes to a point, we obtain a fivefold vertex. This fivefold vertex can decay into threefold vertices in any of five different ways. We are thus confronted with the ugly surprise that the ideal froth, as defined by Smith and by von Neumann, is an incomplete model, for it lacks a prescription for this case. The problem is that there are several different prescriptions that can be argued for with some degree of logic.

The main alternative is whether the process should be deterministic or stochastic. While stochastic decays might make sense in grain growth where thermal fluctuations drive the motion in the first place, they do not quite fit with the ideal soap froth. The reason is the following. One can argue for stochastic decay in an overall deterministic model (when all walls are, all the time, minimal surfaces) if the decaying configuration is an unstable equilibrium, in which case any impurity or imperfection will cause the decay to occur. However, within soap froth, a fivefold vertex (neglecting totally symmetric cases which have zero probability) is not an unstable equilibrium configuration because it is actually not an equilibrium configuration at all. An infinitesimal decay of the vertex, of extent \( \epsilon \), reduces the length of the network by an amount which is linear, and not quadratic, in \( \epsilon \). The question is then which of the five possible decays produces, for the same infinitesimal \( \epsilon \), the largest decrease in length. It is that decay which is obtained by collapsing the walls along the two smallest angles; these
cannot be adjacent, and so the solution is always well defined. Therefore, a fivefold vertex transfers $-1$ unit of topological charge to those two neighbours whose impinging angles are smallest and $+1$ unit to the neighbour in between. Since the impinging angles are necessarily related to the local geometry, it is clear that the process will have a definite polarity. In general, small impinging angles are going to be associated with small bubbles.

### 4.5. The flattening operator

By assuming the existence of a unique scaling state at the ‘microscopic’ level we can derive equations for the background metric and show that the flat Euclidean metric is actually stable. In other words, assuming that there is such a thing as a scaling state, we can study the (very-coarse-grained) evolution of a froth whose initial condition lies everywhere near a scaling state with a slowly varying local lengthscale; we can show that in this case, the variations in local length scale are affected by three different factors:

(a) the von Neumann law;
(b) non-zero mean charges (necessary to achieve the variations in lengthscale);
(c) correlations in the way that topological charge is redistributed after vanishings.

Both (a) and (c) tend to even out the differences in length scale, while (b) tends to destabilize the system. We shall show that (a) prevails over (b).

Assume one such state; call $a$ the average local area as in §4.3. Simple scaling state evolution takes the form

$$\dot{a} = ka,$$

where $\sigma$ was computed by Mullins (1988) to be

$$\sigma = \frac{2\pi}{3} \left\langle a^2 \right\rangle \sum_n \xi_n (n - 6),$$

where $\xi_n$ is the fraction of the total area occupied by $n$-sided bubbles.

Variations in length scale introduce average topological charges which have to be taken care of through eqn. (8). This term introduces, apparently, a destabilization:

$$\dot{a} = k\sigma - \frac{3k}{\pi} a (\ln a).$$

If we write $a = \bar{a}(1 + \epsilon)$ and linearize in $\epsilon$ we obtain a diffusion equation with negative surface tension:

$$\dot{\epsilon} = -\frac{3k}{\pi} \Delta \epsilon.$$

Growth exponents are negative for sufficiently long wavelengths, because of the equilibration action of the drift term, but the mean topological charge associated to variations in $a$ destabilizes sufficiently short length scales. Calling the divide $l_{\min}$, and defining $t = (4\bar{a}/\pi)^{1/2}$, we obtain

$$\frac{l_{\min}}{t} = \left(\frac{3}{4\sigma}\right)^{1/2}.$$
and, since $\sigma$ is of order unity, this means that length scales above a couple of bubbles are already being flattened and hence that topological charge is almost neutral even on small patches. This rigorously justifies the concept of 'topological charge shielding' and shows that the screening length is of the order of one to two bubbles.

Let us note that, according to the preceding lowest-order calculation, the relaxation to a flat global metric is algebraic rather than exponential. Even so, the evolution is such that the long wavelengths are being flattened.

There is one further phenomenon to be considered. If there are two scaling states with different mean lengths meeting at a very narrow interface, this interface will in time move in the direction of the smaller bubbles. This occurs because, as we mentioned above, when five-sided bubbles shrink near the interface, they always transfer one unit of negative charge to the neighbour whose wall is the smallest; in this situation, this neighbour tends to be the smallest neighbour, and hence negative topological charge crosses the boundary in the direction of the smaller bubbles; this accelerates their growth rate, which in turn means that they grow faster until they match the larger bubbles, and so the interface moves.

This effect has been extensively studied experimentally (Glazier et al. 1987, Babcock et al. 1990, Seshadri and Westervelt 1991). Incorporating this phenomenon into our evolution equation we have

$$\dot{a} = k\sigma - \frac{3ka}{\pi} \Delta(\ln a) + \eta a|\nabla(\ln a)|^2.$$  

We see that the last term does not substantially contribute to long wavelengths, for it is only quadratic in the amplitude of perturbations. However, we can see that it has a very dramatic effect on sufficiently short wavelengths, or for sufficiently large area gradients, such as those occurring at an interface between two distinct sizes; in these cases, it can compete in amplitude with the destabilizing middle term.

Therefore, we find that the inclusion of this term does little by way of global flattening when the metric is already nearly flat, but it is very effective for the elimination of large discontinuities in the metric.

§ 5. Conclusions

We can now state what the formulation of a theory of soap froths and related systems look like. The basic ingredients are as follows.

(1) $T$: the triangulation. (Through the geometric analogy, this is a Riemann surface with a special polyhedral structure.)

(2) $A$: the areas; scalar field on this triangulation.

(3) $\eta$: the embedding; a map from $(T, A) \to \text{arcs}^{\text{link}}(T)$, such that the image is a valid embedding of dual $(T)$, the areas are preserved and angles are $120^\circ$.

(4) $\Gamma$: the set of allowed area fields for $T$; the subset of $\mathcal{B}^T$ which admits an embedding (for each $T$, the domain of $\eta$).

(5) $\partial \Gamma$: the boundary of $\Gamma$; an attempt to embed results in a graph having connectivity greater than three or vertices being assigned to the same point.

(6) $\dot{A} = -R(T)$: the von Neumann law. ($R$ is the curvature of the Riemann surface.) This gives the dynamics of the scalar field.

(7) $D(T, A)$: $\partial T \to \mathcal{F}$; an assignment that tells us how the triangulation is to be changed when the scalar field dynamics have collided with $\partial T$. 

The deep questions in the subject remain the same: is there a unique stable scaling state, and what is its structure? We have covered a modest amount of territory in the quest for an answer. We have written down the formulation of the theory with some care, and then we have asked: what is the structure of the states which are asymptotically allowed? We have shown something which is immediately evident to the eye but nevertheless required proof: the dynamics flatten the global structure of the triangulation, both locally, by removing hierarchically nested structures, and globally, by flattening the coarse-grained structure. The asymptotic states are therefore highly correlated at long distances and behave somewhat like a geometric antiferromagnet. In an antiferromagnet, even though individual sites have a magnetic moment, the sum of all magnetic moments in some region cancels to zero much more accurately than if the spins were random. In cellular patterns, even though bubbles have non-zero topological charges, the sum of these charges over some region cancels to zero much more accurately than for random charge placements or even a random topology.

The deepest beauty of this problem is that it can be stated in terms of elementary geometry; yet the smallest victory against the problem seems to require introduction of rather heavy-duty mathematical machinery. In the process of setting up this machinery, we discover deep connections between this problem and other areas, both in physics (string theory and quantum gravity; renormalization processes in growth dynamics) and in mathematics (planar graph theory, integral and algebraic geometry).

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APPENDIX A

THE TEICHMÜLLER SPACE AND STRESS RELAXATION

The issue which remains most obscure in froth theory is that of stresses and stress relaxation. I have left this discussion out of the main text, because it is not central to that discussion while its complexity is distracting.

In a mature, freely coarsening soap froth, most cells appear to have near-unit aspect ratios and the structure is isotropic. We can then properly study the long-distance properties of such a structure through a metric of the form \( g_{\mu\nu} = a^{-1} \delta_{\mu\nu} \). Such a metric is called 'conformal'. It is easy, however, to generate structures which are not isotropic (fig. A 1). In this figure, a regular hexagonal lattice has been stretched and, quite clearly, the number of bubbles in the shortest path between any two bubbles is not a function of the distance between the bubbles alone, but also of their relative orientation. A stressed foam does not have a 'conformal' metric.

The most general metric is a field of 2 × 2 matrices, symmetric with both eigenvalues positive. (When the metric is conformal, the eigenvalues are degenerate.) These eigenvalues have eigenvectors which locally define some directions. Since such a matrix...
also defines an ellipse, we may think of a metric as a field of ellipses. A conformal metric is a field of circles.

These ellipses can be described in terms of their principal axis (the eigensystem) or, equivalently, in terms of their area and an eccentricity vector (or, rather, an eccentricity director). Because of the way that our visual system works, it is very easy for us to pick up the director visually. When a soap froth experiment has just been started, the flow lines that the foam followed while the experimental cell was being filled are very evident to the eye. However, even before the system coarsens by a factor of two, these flow lines have completely vanished. It should be borne in mind that a scaling state is not reached until the froth coarsens by at least a decade. It is thus clear that stress relaxation is much more rapid than relaxation to the scaling state.

So, soap froth in a cell with fixed boundaries will relax stresses very fast, but it is possible to have a froth sustain stresses when there is a continuous motion or flow. It is then interesting to study what the properties of a non-conformal metric are. There is a sizeable body of theory related to the total or bulk amount of stress present in a metric; it is called Teichmüller theory.

A.1. Non-conformal metrics

Two-dimensional Riemannian manifolds have properties that make them unique. They can be studied with techniques from conformal mapping theory in much depth (Bers 1981).

A conformal diffeomorphism of a manifold can be used to map a metric tensor to some other metric tensor. These two metrics are said to be conformally equivalent. An (infinitesimally small) circle is mapped to another (infinitesimally small) circle through a conformal map.

A diffeomorphism which is not conformal will map (infinitesimally small) circles to ellipses. If there is an upper bound ε on the quotient of the major axis to the minor axis of all such ellipses, then the mapping is said to be ε-quasiconformal or to have dilatation ε. The dilatation of f is called $K(f)$. If $K(f) = 1$, then f is conformal.

Not all metrics are conformally equivalent. Consider a torus obtained through glueing opposite sides of a flat square, as opposed to the torus obtained through glueing a $1 \times 2$ rectangle. The map between both metrics is not conformal (since angles cannot be preserved) but two-quasiconformal.
A.2. The Teichmüller theorem

Conformal equivalence is indeed an equivalence relation. It is then meaningful to ask what is the quotient space of 'metrics' modulo conformal equivalence. This quotient space is called the Teichmüller space (Milnor 1982). It can be proven that all Riemannian metrics of a two-dimensional manifold are conformally equivalent to a metric with constant curvature. If this is so, then the following is clear (but technically somewhat arduous).

**Theorem:** The quotient space of all Riemannian metrics of a compact two-dimensional manifold by conformal equivalence is a finite-dimensional space. If the manifold has genus \( g \), \( b \) boundaries and \( p \) punctures, then its Teichmüller space \( T_{g,b,p} \) is homeomorphic to a ball in \( \mathbb{R}^{6g-6+2b+3p} \).

In other words, the Teichmüller space is firstly the space of degrees of freedom for surgery, and secondly the space of all possible bulk stresses which can be embedded in a surface.

The Teichmüller space can be given a metric, the Teichmüller metric, by using the notion of dilatation. Given two metrics, their distance is defined as half of the logarithm of the minimum dilatation of all maps between them. With this definition, the Teichmüller space is a complete metric space.

The Teichmüller theorem asserts that, given two metrics, there is a map between them which is extremal, that is has the smallest possible dilatation, and that this map is unique and has a special form; it is the composition of a conformal map followed by an affine map (a simple stretching) followed by another conformal map.

The core of the Teichmüller theory is the study of the stresses and strains that can be embedded inside a manifold. For instance, if we think of constructing a torus by gluing the ends of a tube, we may twist the tube before gluing and thus embed a permanent stress into the metric. The full construction is as follows: a flat torus can be constructed by gluing opposite edges of a parallelogram. Since congruent parallelograms give rise to essentially the same torus except for overall scale, we may fix one of the edges to lie on the \( x \) axis and to have unit length, that is to be the interval \([0, 1]\). The rest of the parallelogram is fully defined by the position \( z \) of the tip of the other side starting from the origin. We then glue opposite edges. The torus that we obtain will be stressed or strained according to the value of \( z \) (customarily described as a complex number rather than a two-vector). If \( z = i \), then we have a square torus (relaxed). If \( z = 37i \), then we have a long and skinny (strained) torus. If \( z = 1 + i \) we are twisting the torus by one full turn before gluing a stressed torus.

A.3. The Voevodskii–Shabat theorem

All this discussion extends quite nicely to triangulations. Triangulations can be mapped to Riemannian surfaces. An important question is in what sense can Riemannian surfaces be approximated by triangulations. The algebraic sense has been somewhat elucidated.

**Theorem** (Voevodskii and Shabat 1989): Triangulations are dense in the Teichmüller space. In fact the subset of Teichmüller space corresponding to triangulations is one to one with the set of algebraic numbers.

This tells us that, in the Teichmüller space at least, any Riemannian surface can be approximated arbitrarily well by a triangulation.

However, this does not fully solve the problem, as we shall now see.
A.4. Random triangulations and the Weyl–Petersen metric

The Teichmüller space proved to be an important tool in string theory. The functional integral over all Riemannian metrics can be decomposed into an integral over all conformal maps times an integral over the Teichmüller space (or moduli space, as it is called after some extra identifications). However, this integral over the Teichmüller space has to be performed with a weight factor or metric, representing some sort of functional Jacobian, which is called the Weil–Petersen metric. This metric ascertains, for instance, that there are many more metrics conformally equivalent to a ‘square’ torus than metrics conformal to a torus with a very large aspect ratio or, equivalently, that high levels of stress or strain are rarer than moderate levels. (Arbitrarily large levels of strain or stress are allowed, although ‘rare’.)

Even though we know now that triangulations are dense in the Teichmüller space, it has not yet been shown whether the set of all triangulations generates the Weyl–Petersen metric or not. This is the strongest impediment in the study of string theory through random triangulations, and the reason that all such studies up to now were made for a fixed topology, that of the sphere, whose Teichmüller space is trivial (zero dimensional, since there is only one constant curvature metric, the sphere).

A.5. Stress, topological plastics and rheology

The main problem with stress relaxation is that it is mediated through the fast time scale of soap froth. More specifically, the presence of stress results in anisotropies; these make the average length of a bubble side a function of orientation. Sides which are smaller are more susceptible to the side-swapping process; if there are orientations in which sides are smaller, then side swaps tend to occur along these directions. However, numerical simulations which attempt to take this into account need to keep not only the triangulation and the areas of the bubbles (as in the Fradkov–Schvindlerman–Udler–Beenaker model) but also the length and orientation of sides, and so that cannot be done without a full simulation of soap froth as in the work of Weaire and Kermode (1983, 1984), which are numerically intensive in the extreme.

I shall now describe a simpler model, in which stress is dynamically relaxed. The phase space of the model will consist of triangulations, possibly with some boundary conditions.

Given a triangulation $T$ (possibly under suitable boundary conditions), we can embed it through a Poissonian process as in §4.2; this gives us the coordinates of the vertices, which we shall call $\{x\}(T)$. Given these points, we can perform a Voronoi construction to obtain their Delaunay triangulation $T'$. The model allows as ‘stable states’ all those triangulations such that $T' = T$.

The process described above defines a dynamics or, rather, a ‘decay process’, through which a disallowed triangulation decays to some other triangulation:

$$T_{n+1} = \text{Delaunay}[\{x\}(T_n)].$$

We then repeat the process until we obtain an allowed state. I do not have a proof that such a process inevitably reaches a fixed point (an allowed state) but numerical simulations seem to indicate that it is so, at least for triangulations which are not extremely unsmooth.

If we place points on a strip with periodic boundary conditions on the $x$ axis and fixed boundary conditions on the horizontal edges (a finite cylinder with two boundaries), we have what I shall call a topological plastic. If we start from some nice triangulation, we can twist the edges of the cylinder for a while, and the triangulation
will remain allowed (this is the elastic deformation regime). However, at a certain twist, the solution to the Poisson equations will give rise to a Voronoi construction with an edge of zero length (because the four points involved lie on a single circle). This means that, for a twist slightly above that, the Delaunay triangulation of the points will not be equal to the original triangulation. When we replace the triangulation with the new one, the solution to the Poissonian process changes quite discontinuously; we may undo a little the twist and the triangulation will not change back to the original one. A plastic deformation has occurred; this plastic deformation happened to relieve some of the stress imposed by the twist. If we continue to twist, more and more plastic deformations will occur and, even if we twist the boundaries many times around, the stress remains bounded (fig. A 2).

**APPENDIX B**

**GLOSSARY**

The definitions in this glossary should be taken as rough images, rather than real definitions; they are neither accurate nor complete, but try just to evoke the important ingredients in the real definitions. In most cases, technical caveats apply.

*Covariant derivatives.* An operator that acts upon tensor fields the way that a derivative is supposed to behave, that is, it is linear and satisfies the product rule. *Curvature.* The feature of a surface that makes it impossible to map to the plane without distorting either areas or angles. It is manifested in the non-commutativity of derivatives along different directions or, equivalently, in the fact that a vector which is transported 'parallel to itself' along a closed curve may fail to return to its original orientation. *Defect angle.* The angle by which a vector which has been transported around a loop rotates; the defect angle for infinitesimal loops is used to define the curvature tensor. *Delaunay triangulation.* The dual of the Voronoi construction. Given a set of points in the plane, it is the triangulation of that set which maximizes the minimal internal angle of all triangles.
**Derivative operator.** A covariant derivative which annihilates the metric (there is basically only one such derivative); equivalently, the infinitesimal generator of parallel translations on a surface.

**Euler theorem.** For any embedding of a graph on a surface, the number of vertices, edges and facets of the embedding satisfy $V - E + F = \chi$, where $\chi$ the Euler characteristic of the surface, depends only on the surface. The graph has to 'cover the surface'; if it is a graph of the torus, it has to have at least one edge going around the handle (this is a mild condition on the graph).

**Extrinsic.** Given an object $O$, a target space $T$ and a map $\phi: O \to T$, any property of $T$ which can be projected down to $\phi(O)$ and thus, through $\phi^{-1}$, to $O$ itself. It is opposite to intrinsic. The mean curvature is extrinsic, because its definition requires knowledge of distances in the ambient space; a cylinder has a non-zero mean curvature, but it could not be measured on the cylinder, because a sheet of paper can be rolled into a cylinder without distorting it in the least.

**Flat.** Zero curvature. Partial derivatives commute. Equivalently, that can locally be made of pieces of paper without stretching or tearing. The plane and the cylinder are flat; a flat rectangle with opposite sides identified (the traditional 'periodic boundary conditions') is a flat torus.

**Gauss–Bonnet theorem.** Given a Riemannian manifold, the integral of curvature $(d(\text{area}))$ over the manifold is independent of the metric tensor and only depends upon the topology of the manifold: $\int_M R da = 4\pi(1 - \chi)$.

**Genus.** The 'number of handles' of a surface. It has been shown that all compact orientable surfaces can be classified by genus alone, as far as pure topology is concerned; hence, topologically, there is only one compact connected surface of genus two (a two-handled torus).

**Graph.** A finite collection of vertices, any two of which may or may not be joined by edges.

**Graph embedding.** To embed a graph $G$ in space $X$ means to assign to every vertex in $G$ a unique coordinate in $X$ and to every edge in $G$ a continuous curve in $G$ joining the two corresponding vertices, in such a way that no two curves intersect each other (except at vertices as intended).

**Homeomorphism.** A one-to-one mapping which preserves the topology, that is the definition of open sets. It prevents self-intersections and accumulation points.

**Intrinsic.** Some property which can be defined on some object without any reference to any target or ambient space; it refers exclusively to properties of the object. It is opposite to extrinsic. For example the Riemannian curvature is intrinsic, because it can be measured on the surface without any need to go through any ambient space (through the defect angle, for instance).

**Isometric embedding.** An embedding of a Riemannian manifold into Euclidean space in such way that intrinsic and extrinsic metrics coincide. Isometric embeddings always exist for sufficiently large target spaces (Gromov 1986).

**Isometries.** The symmetries preserving the metric structure. Translations and rotations are isometries of the plane; Möbius maps are the isometries of hyperbolic space. Only manifolds with a high degree of symmetry have non-trivial isometries.

**Kristoffel symbol.** An object used to relate two different covariant derivatives. It has the apparent form of a tensor but it is not a tensor (hence the name 'symbol'). In common usage, it relates the derivative operator to the ordinary partial derivatives of the coordinate system.

**Manifold.** An object which locally looks, everywhere, like Euclidean space, but which
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May have a more complicated global structure. A sphere, a Klein bottle and the set of all orthogonal matrices of unit determinant are manifolds. More generally, a (smooth) real manifold is defined as a space $M$ and a collection of (one-to-one) maps $\phi$ from open sets in the manifold to open sets in Euclidean space, in such a way that this collection of maps covers the manifold and has some form of self-consistency. (The collection is called an atlas.)

Manifold embedding. A map from a manifold to some (necessarily larger) space, preserving the topology (i.e. homeomorphic; equivalently, preserving the 'differentiable structure' or the set of smooth functions). The set of points at unit distance from a centre is one of many possible embeddings of the sphere.

Mean curvature. An extrinsic measure, depending solely on the ambient space. The harmonic mean of the two principal radii of curvature. A cylinder has a non-zero mean curvature even though it can be made of paper without stretching and tearing.

Planar graph. A graph admitting an embedding in the plane. It can be drawn on paper without links crossing each other.

Quotient space. Given a space $S$ and an equivalence relation $\sim$ on $S$, there is a well defined space given by choosing one member of each equivalence class, denoted by $S/\sim$ and called the quotient space. If numbers on the real line are identified whenever their difference is an integer (a wavelength condition), then the quotient space of the reals over this equivalence is a circle (a phase). Quotient spaces are very useful, but 'bad' equivalence relations may give rise to rather untame spaces.

Ricci tensor. A curvature tensor with two indices obtained by taking traces of the Riemann tensor.

Riemann curvature. On a surface, the product of the two principal curvatures at a point. On a Riemann manifold, the trace of the Ricci tensor.

Riemannian manifold. A manifold equipped with a positive definite tensor $g_{ab}$ with two indices, called the metric, and where differential of lengths are defined through $ds^2 = g^{ab} dx_a dx_b$.

Riemannian surface. A space where one can, locally, do complex function theory as in the complex plane. More precisely, a complex one-dimensional manifold, that is a space and a collection of maps as in manifold, but the target space is the complex plane and the maps are required to satisfy analyticity conditions.

Saturated. A saturated graph is one for which all vertices are connected to each other, that is a simplex; a saturated planar graph is one for which no more edges can be added without the graph losing planarity; a saturated graph of a surface is a graph accepting no more edges without losing the property of being embeddable in that surface.

Scalar curvature. Riemann curvature (the trace of the Ricci tensor).

Schwinger–Dyson equations. A field-theoretical technique used to derive the number of diagrams at a given perturbation order in a field theory.

Simplex. Topologically, a graph such that all vertices are connected to each other. Geometrically the set of $x_i \geq 0$ such that $\Sigma x_i = 1$. In either case, it is the $n$-dimensional analogue of a triangle.

Simplicial complex. A higher-dimensional analogue of a triangulation, made by glueing simplices together at their facets.

Surgery. Topological technique consisting of cutting and glueing simple shapes to obtain complex shapes. Cutting is performed by the introduction of boundaries; glueing is achieved through maps which identify those boundaries. If a metric structure is to be
preserved, the maps used to glue should be isometries. Surgery for two-dimensional manifolds is fairly straightforward and was developed by Klein, Poincaré, Hadamard and others in the last century; surgery on higher-dimensional spaces is a rather arcane subject.

**Topological charge.** In a triangulation, the connectivity of a vertex minus 6: $n - 6$. It is called a charge because, by virtue of the Euler theorem, its sum over the triangulation is a constant depending on the surface only.

**Triangulation.** A saturated graph of a surface; a graph for which all the facets are triangles.

**Two-torus.** Siamese doughnuts; the surface of the rims of a pair of eyeglasses.

**Voronoi construction.** This construction (also called 'Breit–Wigner' cells and used in the definition of the 'first Brillouin zone') proceeds as follows. Given a set of points $x_i$ in some metric space $X$, we subdivide $X$ in portions $X_i$ in such a way that all points in each portion are nearer to $x_i$ than to any other $x_j$: $X_i = \{x | \text{dist}(x, x_i) < \text{dist}(x, x_j) \forall j \neq i\}$.

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