Four Sided Domains in Hierarchical Space Dividing Patterns

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The cracks observed in the glaze of ceramics form networks, which divide the 2D plane into domains. It is shown that, on the average, the number of sides of these domains is four. This contrasts with the usual 2D space divisions observed in Voronoi tessellation or 2D soap froths. In the latter networks, the number of sides of a domain coincides with the number of its neighbors, which, according to Euler's theorem, has to be six on average. The four sided property observed in cracks is the result of a formation process which can be understood as the successive divisions of domains with no later reorganization. It is generic for all networks having such hierarchical construction rules. We introduce a "geometrical charge," analogous to Euler's topological charge, as the difference from four of the number of sides of a domain. It is preserved during the pattern formation of the crack pattern.

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In a two-dimensional space-dividing network, a set of lines forms a reticulum dividing a surface into adjacent domains. Very different mathematical, physical, biological, or social processes give rise to such networks. Examples include the Voronoi tessellation of randomly distributed points, the two-dimensional soap froths confined between glass plates, the reticulum formed by cracks in thin layers, the cellular structure of 2D living tissue, the leaf venation, or the division of geographical surface by roads, streets, or field borders. Depending on the specific example, elementary domains correspond to areoles, blocks, or fields, while line segments which delimit them can be walls, cracks, veins, streets, or hedges. The points where these lines meet are called vertices and generically, a vertex is the meeting point of three lines. These networks have a well known common property expressed by Euler's theorem which states that in the plane each domain has, on average, six neighbors ($\langle n \rangle = 6$). For many cellular networks, it follows that the average number of sides per cell, $\langle s \rangle$, is also six. This property is crucial for the understanding of two-dimensional soap froth which is a model system for this kind of pattern [1,2].

The present Letter aims at demonstrating the existence of a subset of these networks that we will call hierarchical reticula, where geometry imposes that the domains are, on average, four sided. While compatible with Euler's theorem, the average of four sides is the signature of this hierarchy. It is the consequence of a formation process that can be described as the successive divisions of domains and the absence of any further reorganization.

Let us first consider the crack reticulum formed during cooling in the superficial glaze of ceramic plates as shown in Fig. 1. This cracking pattern is due to differential shrinkage; similar patterns are also observed in thin layers of desiccating mud or gel [3,4]. In these systems, either cooling or desiccation induces a shrinkage of a thin material layer, which is frustrated by adhesion to a rigid substrate. The resulting mechanical stresses are partly released by the fractures. It has been shown that the characteristic distance between the fractures scales linearly with the layer thickness. The studies in colloidal material have furthermore revealed the existence of two distinct cracking regimes. In the case of very thin layers or inhomogeneous materials, the fractures are almost simultaneously nucleated in starlike triplets and show a very rough appearance. The angles at the nucleation points are mostly 120°. Numerical models [5,6] have been employed to discuss the scaling behavior of this regime. This approach is particularly pertinent when compared with experiments in granular materials [7].

The cracking regime we are interested in is observed for thicker layers, or in brittle, homogeneous materials such as glaze. Here, the nucleation of cracks is scarce and the fractures are formed successively. In general, no more than one crack is propagating at a given time. When the extremity of a propagating fracture comes in the vicinity of an older, already formed one, it will propagate to join the older one by a right angle (principle of local symmetry) [3,4]. The new crack, clearly, does not affect the position of

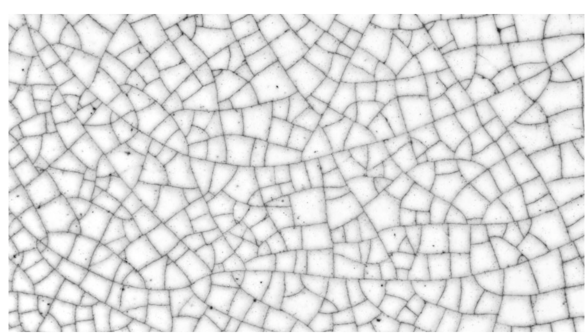

FIG. 1. A hierarchical crack pattern in the glaze of a ceramic plate. The first crack to have formed can be seen as a long sinuous line along the lower part of the photograph.

the older ones. We thus understand the crack pattern geometry as the result of successive domain divisions without reorganization. A fracture divides the domain in which it is formed into two. As the domains decrease in size, the fractures decrease in length. The frozen hierarchy of cracks of different lengths is thus the signature of their successive formation. In the case of colloidal materials, the crack opening increases in time and the crack succession is also manifest in the crack width [8,9].

The way a domain is divided by a new crack will depend on the mechanical stress field and thus on the domain shape, which defines the boundary conditions (see also the phenomenological modeling in [10]).

The pattern in Fig. 1 is composed of triangles, quadrangles, or pentagons. In order to put this observation on a quantitative basis we counted the number of sides of $N = 1000$ domains on such plates (histogram in Fig. 2). The average number of sides is very close to four. This result is in sharp contrast to what is found for cellular patterns such as soap froths (Fig. 3) where the average number of sides is six.

The different average number of sides can be understood by considering the two details of Fig. 3. We see that it is necessary to make a clear distinction between topology and geometry. Edges and vertices belong to the topological description of the space-dividing network. In the 2D foam, an edge is a soap film separating two bubbles and a vertex is the point where three films meet. In a hierarchical crack pattern, a vertex is the point where a younger crack joins an older crack and an edge is the part of the fracture between two of these points. An edge separates two adjacent domains. The number of edges delimiting a cell is thus equal to the number of its first neighbors.

By contrast, we have to consider ‘‘sides’’ and ‘‘wedges’’ (or ‘‘corners’’) if we wish to describe the shape of a cell. A side of a domain is the part of the contour delimited by two wedge-shaped singularities, the corners of the cell. A side can be curved but its curvature is continuous. It is clear that in 2D the number of corners of a cell is equal to the number of sides, while the number of vertices on the cell contour is equal to the number of delimiting edges. However, the relation between edges and sides depends on whether the network is hierarchical or not. Let us now revisit the well-studied case of the foam. Here, the angles between two edges is 120° (Plateau’s law), each vertex corresponds thus to a corner. In this case, the number of sides is equal to the number of the delimiting edges and thus neighbors. The marked foam cell in Fig. 3(a) has six sides and six neighbors.

Considering the foam pattern as an embedded graph, Euler’s theorem on topology can be applied. This theorem states that for a connected graph, the total number of vertices V, edges E, and separated cells N are related by

$$N - E + V = O(1). \tag{1}$$

The constant term of the order of one, O(1), depends on the topology of the embedding space and on the boundary conditions. It has been shown that in a 2D soap foam, only threefold vertices are stable. The total numbers of edges and vertices are therefore related by $3V = 2E$. As stated above, each edge presents a side for each adjoining cell. The total number of sides S is thus $S = 2E$. Consequently, the average number of sides $\langle s \rangle$ is

$$\langle s \rangle = S/N = 6[1 - O(1)/N]. \tag{2}$$

In the case of an extended pattern with a large number of cells N, the average number of sides (and the average number of neighbors) is thus equal to six. Let us furthermore note that it has been found useful to introduce the notion of a topological charge of a cell by

$$q_{\text{topo}} = 6 - n, \tag{3}$$

where n is the number of neighbors of that cell. Local topological transformation observed in a soap froth are the neighbor switching (T1) and the vanishing of a cell (T2). Both conserve the total topological charge of the

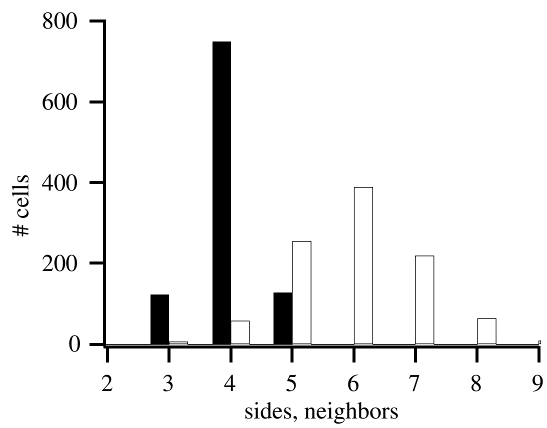

FIG. 2. Histograms for $N = 1000$ domains in ceramic plates of the number of sides (filled bars) and of the number of neighbors (empty bars). The average number of sides is $\langle s \rangle = 4.007$; the average number of neighbors is $\langle n \rangle = 5.98$.

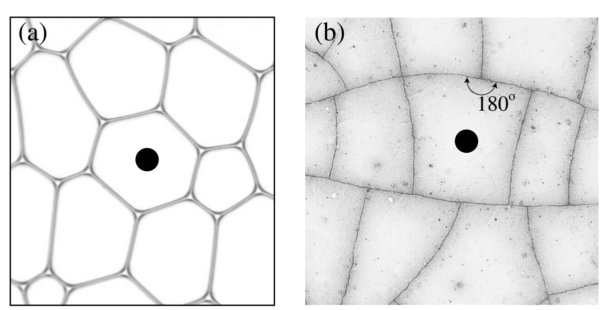

FIG. 3. (a) A detail of a dry, two-dimensional foam. The cell marked by the disk has six sides and six neighbors. (b) A detail of a hierarchical crack pattern. The marked cell has fours sides, but six neighbors.

involved cells so that the use of the term charge is justified (see also [11]).

In the case of a domain bounded by cracks [Fig. 3(b)], the number of sides is not equal to the number of neighbors. When a crack joins the contour from the outside, it forms a vertex with a 180° angle inside the domain (because the older crack has not been disturbed). These vertices thus do not form wedges in the contour. The marked crack domain in Fig. 3(b) has four sides, but six neighbors. The inequality of the numbers of sides and delimiting edges is thus the direct result of the frozen hierarchy of the crack pattern.

Since at each crack vertex one of the three angles is equal to 180°, the average number of sides $\langle s \rangle$ should relate to the average number $\langle n \rangle$ of neighbors by $\langle s \rangle = 2/3\langle n \rangle$ and thus has to be four. Let us emphasize that the four sided domains are the consequence of the 180° angle and not of the right angle between the newer and the older crack. The average number of neighbors, however, is still constrained by Euler to be six (see Fig. 2).

A direct demonstration can be derived from the dynamics of formation of the crack pattern, i.e., from the successive division of the domains. As illustrated in the sketch in Fig. 4, a four sided domain can be divided either into two four sided domains or into a triangle and a pentagon. A triangle can only be divided into a triangle and a quadrangle, while a pentagon can be divided into a quadrangle and a pentagon, or into a hexagon and a triangle. The crack presents a new side for each daughter domain ($+2$) and divides two sides of the mother domain (again $+2$). The number of sides of the "daughter" domains s_a and s_b are thus related to the number of sides of the "mother" domain s by

$$s_a + s_b = s + 4. \tag{4}$$

We excluded the situation where the crack ends in a corner of the mother domain. This nongeneric case is neither stable under perturbations nor observed in real crack patterns: the stress is minimal in a corner and the propagating crack will avoid this region. Let us now, in analogy to the topological charge, introduce a *geometrical* charge q_{geo} of a s-sided cell as

$$q_{\mathrm{geo}} = 4 - s. \tag{5}$$

Equation (4) can then be written as a conservation law:

$$q_{\mathrm{geo},a} + q_{\mathrm{geo},b} = q_{\mathrm{geo}}, \tag{6}$$

the sum of the geometrical charges of the daughter cells is equal to the geometrical charge of the mother cell. This conservation law justifies the term charge. In the extended crack pattern, the average number of sides can be written in terms of the total geometrical charge $Q_{\mathrm{geo}} = \sum q_{\mathrm{geo},i}$:

$$\langle s \rangle = \frac{1}{N} \sum s_i = 4 - \frac{Q_{\mathrm{geo}}}{N}. \tag{7}$$

Since the total geometrical charge is conserved during the successive divisions, it is equal to the geometrical charge of the initial cell, and thus of the order of one. If the initial cell is a quadrangle ($q_{\mathrm{geo}} = 0$), the average number of sides is strictly four, otherwise it will converge rapidly to four as $1/N$.

The geometrical charges show a particular dynamic, which is linked to the fact that the daughter cells become completely independent. Once a pair of charges (a dipole) is created, it can never annihilate. Both charges will "propagate" (and transform) independently. Figure 5 shows a example of the propagation of $q_{\mathrm{geo}} = -1$ defect during the cell divisions. The charge becomes more and more localized, and, since the number of cells increases in time, diluted. During the following divisions, other pairs of charges are created. It is not clear if the statistical distribution of the charges (or the number of sides) is conserved during the formation process, i.e., if the successive cell division is self-similar. This lack of self-similarity would be complementary to the observed scaling of the crack width [9].

The crack pattern is a model for a hierarchical space division observed in other systems. We can now examine briefly two other examples. A detail of the venation pattern of a plant leaf is shown on Fig. 6(a). The veins form a

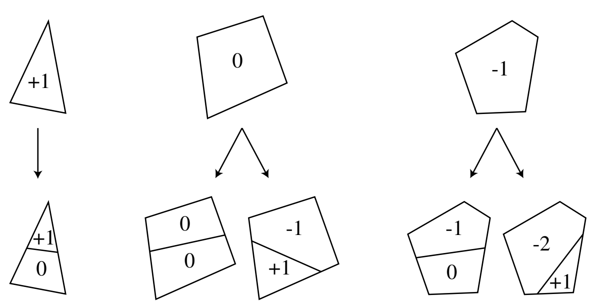

FIG. 4. The possible divisions of a triangle, a quadrangle, and a pentagon. The numbers indicate the geometrical charges q_{geo} of the shapes.

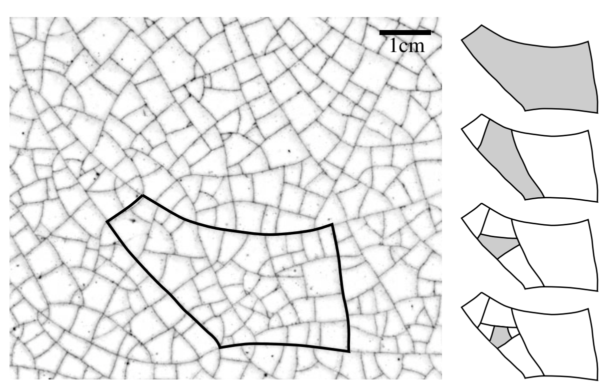

FIG. 5. The propagation from large scale to small scale of an $q_{\mathrm{geo}} = -1$ defect.

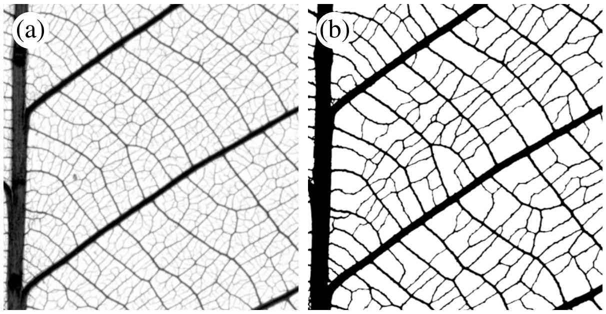

FIG. 6. The venation network of a vegetal leaf. (a) A photograph of the venation pattern. (b) The smaller veins and the free dangling veinlets are removed, using a image processing described in [13].

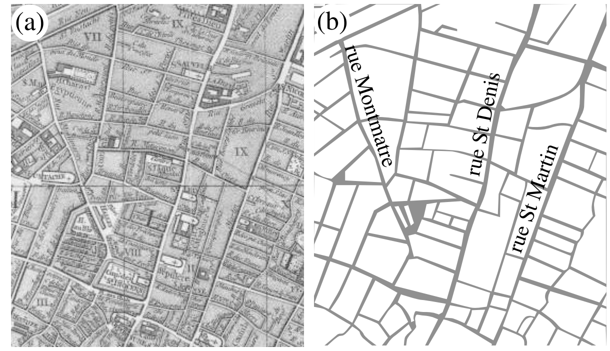

FIG. 7. Paris 1760. (a) Map drawn by Robert de Vaugondy, Didier (1723–1786). (b) Redrawn by the authors.

hierarchical system in which the veins form successively during the leaf growth. When a new vein forms it connects to older ones at both its ends. As it is the case for the crack pattern in colloidal materials [9], the thickness of the veins can be considered as the indicator of their age. A hierarchical reticulum is thus formed with the same type of construction as a crack pattern [12]. In the large scale structure [Fig. 6(b)], the dominance of the four sided domains is observed. However, the measurement of the angles at the vertices reveals that angles close to 180 ° are only observed when a very thin vein joins a large one [13]. Depending on the relative thicknesses of the veins meeting at a branching point, the angles between the two vein segments belonging to the large vein vary between 120 ° and 180 °, presumably because there is a partial reorganization of the veins after their formation. The criteria for singularities corners in the cell contour become more subjective, and so does the counting of sides. We can now turn to a network generated by human activity. Figure 7 shows an image-processed detail of a city map of Paris in 1760. The names of the longer streets indicate that they were originally roads leading from the center of Paris to neighboring villages or abbeys; they were there first. Some of the transverse streets were probably formed as ways for the carts to reach the fields. With the increase of population density, new streets became necessary, resulting in the observed structure. There are several noticeable features. Most of the vertices are threefold and correspond to the meeting of a new street with an older one. This means that the division of a block by a street was usually independent from the division of other blocks. As in cracks the connection of a new street has no influence on the shape of the old ones so that most vertices have one 180 ° angle. The street network has thus the characteristic of a hierarchical reticulum and the four sided blocks dominate. We chose a map where the city growth resulted from self organization. With urban planning, the street networks are globally decided at once so that the structures are different with, e.g., checkerboard structures and fourfold (or more) vertices.

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